Transpositionally Invariant Subsets: A New Set-subcomplex

by

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Since the appearance of Allen Forte's two seminal studies in pitch-class set theory (Forte 1964 and 1973), considerable effort has gone into the development of tools for measuring the relatedness of set-classes. Most of the proposed tools, Forte's among them, are founded on criteria of either inclusion or shared interval content.

Despite the proliferation of the latter type of relation over the last decade or so,[1] the former type—based on inclusion, or the subset/superset relation—retains a musically intuitive appeal: on the one hand, the capability of determining the relatedness of any pair of sets according to a universal, abstract scale may be both feasible and useful; on the other, the observation of relatedness by inclusion has an undeniable immediacy that can be expressed compositionally in multiple, audible ways.

The set-relations described here were first introduced in a paper presented at the annual meeting of the Society for Music Theory in Ann Arbor, Michigan, in November 1982.

The author wishes to express his gratitude to John Clough, who provided Theorem 4 as well as many helpful suggestions for revision.

[1]See, for example, Teitelbaum (1965); Morris (1979/80); Rahn (1979/80); Lewin (1979/80); and Isaacson (1990). Forte's (1973) R-relations also belong to this category, although they are limited to sets of equal cardinality. The reader is referred to Isaacson's article for a useful summary of the previous developments in this area.

This immediacy provides a compelling rationale for Forte's reliance on the set-complex K in his theory and analyses. Early on, however, Forte determined that the set-complex is a measure of relatedness insufficiently fine to stand alone; he observes, in fact, that it was his dissatisfaction with the sheer size of the set-complex that motivated his development of a set-subcomplex.[2] After all, if a given relationship holds for a large number—even a majority—of sets, the observation of this relationship loses much of its analytical power. The criterion for membership in the subcomplex Kh, which Forte calls the "reciprocal complement relation,"[3] is a relationship considerably more exclusive than the simple inclusion relation, which determines membership in the set-complex K. This restrictiveness is reflected in the sizes of the sub-complexes Kh: for example, the set-complex K about sets 6-Z19/Z-44,[4] which is illustrated on the cover of *The Structure of Atonal Music*, includes thirty-seven sets and their complements, while the subcomplex Kh about the same pair of sets includes only fifteen sets and their complements.[5]

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[2]Forte (1973), p. 96. All non-dated references to Forte in the present paper refer to this work.

[3]Ibid.

[4]For ease of reference, I have adopted Forte's set names and, with one minor variation, his notations for pc sets and interval vectors: a set will be displayed in parentheses, with the pc integers set off by commas; an interval vector will be enclosed in square brackets.

[5]This count follows Forte's convention of considering only sets of cardinality 3 to 9, inclusive. The subcomplex Ks, introduced in Forte (1964) and subsequently abandoned, used a different mechanism to determine membership, but was intended to accomplish more or less the same purpose.

Forte's most recent work in this area (Forte 1988) similarly uses a restricted form of the inclusion relation to construct set groupings which he calls "genera." The mechanism used here is a restriction I call the "continuous sequence requirement": that is, in order for set B of cardinal N = 5 or 6 to be a member of a genus about a given trichord A, not only must B and its complement be supersets of A, but B must also be a superset of some genus member of cardinal N – 1.[6]

Another qualified form of the inclusion relation, that in which a subset of cardinal N – 1 of a given pc set of cardinal N is held invariant under transposition of that set, is treated briefly in Forte (1973), and forms the basis of the ideas to be developed here.[7] As Forte observes, this property is limited to a relatively small number of sets.[8] The present article details the abstract relationships of these sets, proposes a new theoretic construct based on these relationships, and suggests some potential applications of this construct to musical composition and analysis.

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[6]Forte formulates his Rule 2 as follows: "each pentachord must contain at least one of the tetrachords in the genus and each hexachord must contain at least one of the pentachords and at least one of the tetrachords in the genus." (Forte 1988, p. 192) Perceptive readers will notice that the final stipulation is actually unnecessary because of the transitive property of the inclusion relation.

[7]The relationships discussed below generally pertain to set-classes, or collections of unordered pc sets equivalent under transposition and/or inversion. The more general term "set" is to be understood as referring to a set-class, or to its prime-form representation, unless explicit reference is made to a particular collection.

[8]Forte (1973), p. 37. The relationship becomes far more common, and its occurrence therefore arguably less significant, as the difference between cardinalities increases.

Definitions and Theorems

Most of the basic principles underlying the construction of the subcomplex introduced here are fundamental to pitch-class set theory, and can be found in Forte (1973), Rahn (1980), or Morris (1987); for the convenience of the reader, and for the sake of formal consistency, they are summarized in the following definitions and theorems.

> THEOREM 1: The number of pcs that remain invariant when a set is transposed is equal to the nth entry in that set's interval vector, where n = t or 12 − t, and t is the interval of transposition in semitones. Where n = 6, the number of invariants is equal to twice the vector entry.
>
> COROLLARY 1.1: Any set of cardinality N whose interval vector includes an nth entry m such that m = N (or, for n = 6, such that m = N/2) will be totally invariant when transposed with t or 12 − t = n.[9]

Thus, for example, set 6-20 (0,1,4,5,8,9), with vector [303630], will yield 3 invariant pcs under transposition with t = 1, 3, or 5 (or 11, 9, or 7) semitones; however, for t = 4 or 8, it becomes (4,5,8,9,0,1) or (8,9,0,1,4,5), in each case transposing completely into itself.

Table 1 is a list of all sets of cardinality 2 through 10 that remain totally invariant under transposition, along with their prime forms and interval vectors; for each set, the vector entry corresponding to a transposition yielding total invariance is underscored. For some sets there is more than one such entry.

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[9]For these and the theorems that follow, values for ic6 must be equal to half those of other ics because of the tritone's well-known property of mapping onto itself under complementation.

Table 1. Totally invariant sets

2-6	(0,6)	$[00000\underline{1}]$
3-12	(0,4,8)	$[000\underline{3}00]$
4-9	(0,1,6,7)	$[20002\underline{2}]$
4-25	(0,2,6,8)	$[02020\underline{2}]$
4-28	(0,3,6,9)	$[00\underline{4}00\underline{2}]$
6-7	(0,1,2,6,7,8)	$[42024\underline{3}]$
6-20	(0,1,4,5,8,9)	$[303\underline{6}30]$
6-30	(0,1,3,6,7,9)	$[22422\underline{3}]$
6-35	(0,2,4,6,8,10)	$[0\underline{6}0\underline{6}0\underline{3}]$
8-9	(0,1,2,3,6,7,8,9)	$[64446\underline{4}]$
8-25	(0,1,2,4,6,7,8,10)	$[46464\underline{4}]$
8-28	(0,1,3,4,6,7,9,10)	$[44\underline{8}44\underline{4}]$
9-12	(0,1,2,4,5,6,8,9,10)	$[666\underline{9}63]$
10-6	(0,1,2,3,4,6,7,8,9,10)	$[88888\underline{5}]$

> THEOREM 2: The complement of any set that remains totally invariant under transposition will also remain totally invariant for the same values(s) of t.[10]

This result can be derived from Theorem 1 in conjunction with the following theorem:

> THEOREM 3: The interval vector of any set of cardinal $N > 6$ can be computed by adding the constant $2(N - 6)$ to each vector entry of its complement; for ic6, half this value, or $N - 6$, is added.

Thus, for each pair of complementary cardinalities (i.e. cardinalities that sum to 12), there is a constant "complementation vector" that, when added to the vector of any set of cardinal $N < 6$, yields the vector of its complement. The complementation vectors are as follows:

[222221] for $N = 5, 7$

[444442] for $N = 4, 8$

[666663] for $N = 3, 9$

[888884] for $N = 2, 10$.[11]

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[10]The list of totally invariant sets in Forte (1973), p. 37, omits sets 2-6, 3-12 and 6-35, and makes no mention of the complementary sets of cardinal 8, 9 and 10. Forte (1964) does include these complementary sets in a table of sets "without distinct I-forms and with fewer than 12 distinct transpositions," the latter being another way of describing total transpositional invariance. This table also includes the three sets missing from Forte (1973); however, it correctly omits 6-30, the only set that holds the second but not the first of these properties.

[11]Thus, for example, to calculate the interval vector for set 8-25, we can first determine the vector of its complement, 4-25, and then add the complementation vector [444442]:

4-25		[020202]
	+	[444442]
8-25		[464644]

Since the constant $2(N - 6)$ representing an entry in the complementation vector for a given pair of cardinalities is equal to the difference between those cardinalities (or, for ic6, half of that value), for any vector entry yielding total invariance, there will be a corresponding entry in the vector of the complementary set that likewise yields total invariance.[12]

> DEFINITION 1: Given two sets A and B, B is a transpositionally invariant subset (TIS) of A iff:
>
> a) $B \subset A$, and
>
> b) $B \subset T_t(A)$, where $t \neq 0$.

This relationship is both most exclusive and most perceptible when the cardinalities of the two sets differ by only one. Sets of cardinality N holding $N - 1$ cardinality subsets can also be identified through inspection of the interval vector:

> COROLLARY 1.2: Any set A of cardinal N whose interval vector includes an nth entry m such that $m = N - 1$ (or, for $n = 6$, such that $m = (N - 1)/2$) will hold a subset B of cardinal $N - 1$ invariant when transposed with t or $12 - t = n$.

Notice that in each of these sets the entry for ic6 indicates that the set will remain totally invariant when transposed with $t = 6$. Similar results obtain for complementary sets of other cardinalities.

These vectors have been noted in a number of sources, including Forte (1973) and Forte (1964), where they are referred to as "difference vectors." They were first pointed out, I believe, in Hanson (1960), pp. 351–52. Note that the formula given in Theorem 3 yields the complementation vector [000000] for $N = 6$, demonstrating that all (nonequivalent) complementary hexachord pairs are necessarily Z-related.

[12]See the illustration given in note 11, above.

This, of course, is another consequence of Theorem 1. For example, set 5-31 (0,1,3,6,9), with vector [114112], yields the invariant subset 4-28 (0,3,6,9) under transposition with t = 3, 6, or 9. While it is necessarily also true that the given 4-28 subset, when transposed by any of these values of t, remains a subset of the untransposed 5-31, this is not the same as saying that two or more <u>different</u> 4-28 sets are subsets of a given form of 5-31; since 4-28 is itself totally invariant, it transposes into itself with t = 3, 6, or 9. Rather, given a particular form of 5-31 that includes a particular 4-28 subset, the inclusion relation can be preserved under appropriate transposition of either set.

Forte (1973) gives a chart showing invariant subsets of cardinal $N - 1$ for sets of cardinal N, where $4 \leq N \leq 8$.[13] As I will show, it is instructive to be able to view the occurrences of this relation across the full range of cardinalities; accordingly, an expanded version of Forte's chart, incorporating sets of cardinal 2 through 10 (for $N = 10$ and for $N - 1 = 2$) is given in Table 2.[14] This table additionally incorporates three sets omitted by Forte although they fall within his specified range of cardinalities (7-28, 5-15, and 5-19), and alters the display to eliminate multiple listing of sets. All invariant subset relationships are indicated in Table 2 by lines connecting the set names.

[13]Example 42, p. 38.

[14]Sets of cardinal 0, 1, 11 and 12 are omitted as uninteresting, there being only one set-class for each of these cardinalities.

Table 2. Invariant subsets of cardinal N - 1 for sets of cardinal N.

10-1 —	9-1 —	8-1 —	7-1 —	6-1 —	5-1 —	4-1 —	3-1 —	2-1
10-5 —	9-9 —	8-23 —	7-35 —	6-32 —	5-35 —	4-23 —	3-9 —	2-5
10-3 —	9-10 —	8-28						
10-4 —	9-12							
10-2 —	9-6 —	8-21						
			7-33 —	6-35				
		8-24						
	9-5 —	8-9						
	9-8 —	8-25						
		8-19 —	7-21 —	6-20				
			7-7					
				6-7				
			7-15					
			7-28					
			7-19 —	6-30				
			7-31					
				6-27 —	5-31 —	4-28		
					5-15			
					5-28 —	4-25		
					5-33 —	4-21 —	3-6 —	2-2
						4-24		
							3-12	
					5-21 —	4-19		
					5-7			
						4-9		
					5-19			
							3-5	
							3-8 —	2-6
							3-10	
								2-3

Forte's statement that the TIS relation is not a transitive one[15] is evidently based on a restricted concept of this relation, which might be expressed by adding the following stipulation to Definition 1:

c) There is no set S such that $S \subset A$, $S \subset T_t(A)$, and $B \subset S$.

Thus, for example, Forte would not consider 4-28 a TIS of 7-31, since 4-28 is a subset of 6-27, which is a TIS of 7-31. This restriction neatly circumvents the problem that the TIS relationship given in Definition 1 becomes far less exclusive for pairs of sets whose cardinalities differ by more than one; for example, set 8-19 holds invariant a majority of 4-element sets, and all twelve 3-element sets. However, the same result can be obtained by invoking the continuous sequence requirement to construct tuples from the contents of Table 2.

> DEFINITION 2: A transpositionally invariant subset (TIS) tuple is a group of sets whose cardinalities form a continuous sequence N, N − 1, N − 2, etc., with one and only one set for each cardinality represented, in which every set is a transpositionally invariant subset of the next larger set.

Each such tuple is represented in Table 2 as a chain of sets connected by lines, progressing unidirectionally from column to column. Thus, {5-33, 4-24, 3-12} is a TIS tuple, while {5-28, 4-25, 5-15} is not.

[15]Forte (1973), p. 37.

Special Properties of Transpositionally Invariant Subset Tuples

The tuples shown in Table 2 display a number of remarkable properties and relationships. Sets holding transpositionally invariant subsets are "saturated" with at least one interval class;[16] accordingly, many of the tuples can be visualized as being constructed of repeated superimpositions of a single ic.[17] The two M5-related tuples beginning with 2-1 and 2-5 (reading, for the moment, from right to left) demonstrate this property most clearly, being built up by the accretion of ic1 and ic5, respectively. It should now be clear why these are the only two TIS tuples that are continuous over the entire range of cardinalities: 1 and 5, being prime to 12, yield all 12 pcs through repeated superimposition. By contrast, 2, 3, 4, and 6, being factors of 12, produce pc replications without achieving the total chromatic. For example, the accretion of ic2 produces the tuple {5-33, 4-21, 3-6, 2-2} (read right to left); however, further addition of ic2 produces 6-35, or the whole-tone scale, which, since it is totally invariant, is not included in the tuple. The tuple {10-2, 9-6, 8-21, 7-33, 6-35}, containing the

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[16]That is, these sets have at least one maximally large entry in their interval vectors. They are therefore the same as Eriksson's (1986) "maxpoints."

[17]This is the simplest instance of "transpositional combination" as developed by Cohn (1988). Cohn's method further conceives many larger symmetrical sets by means of combining a given set with its transposition, that is, by transposing the interval(s) of a set through the intervals(s) of second set: e.g., 3-12 (0,4,8) * 2-1 (0,1) = 6-20 (0,1,4,5,8,9). The reader may also be reminded of Hanson's concept of "projection." This concept provided an elegant means of deriving many of the more homogeneous sets; however, attempts to apply it to more complex constructions, while normally successful, appear more and more tortuous.

complements of these sets, is built up by the addition of members of the second whole-tone collection; the point of discontinuity between the two tuples is the point at which the first whole-tone collection is exhausted.

As exemplified by the foregoing illustrations, many of the sets found in Table 2 are familiar as chords or collections. Composers have found particularly useful sets that feature a high degree of symmetry or are exceptionally rich in one or two intervals; conversely, certain constructions may be useful precisely because of the intervals they lack. The whole-tone scale (6-35) is one set that suggests such an interpretation.[18] Among the other familiar constructions represented in Table 2 are: the various chromatic segments (beginning with 2-1, as observed above); the pentatonic and diatonic collections (5-35 and 7-35); the octatonic collection and its complement, the diminished seventh chord (8-28 and 4-28); the augmented and diminished triads (3-12 and 3-10); the so-called Z-cell (4-9);[19] and all of the incomplete whole-tone collections (5-33 and its subsets).[20]

As indicated in Theorem 2 above, the complement of any totally invariant set will also be totally invariant; similarly, and again as a consequence of Theorem 3, the complement of a set that has an invariant subset of cardinal N − 1 will also have such a

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[18]Clough (1983) offers an ingenious method of characterizing sets according to the criterion of exclusion.

[19]This set was first identified as "cell Z" in Treitler (1959).

[20]See Mead (1984) for a useful method of categorizing and manipulating sets based on their whole-tone subsets.

subset. Less intuitively obvious is the fact that if a set is present in Table 2, as either a subset or a superset, its complement will also be present.[21]

We can understand why the population of Table 2 consists entirely of complementary pairs by observing that all of the sets fall into one of two categories: they are either sets with invariant subsets, which occur as complementary pairs as described above, or ends (i.e., smallest sets) of TIS tuples. But the ends of tuples are, without exception, totally invariant sets (the sets listed in Table 1, and their complements); conversely, all totally invariant sets appear as ends of tuples in Table 2. Since these sets likewise occur as complementary pairs (see Theorem 2), every set that appears in Table 2 is accompanied by its complement.[22]

A final complementation-based relation resides in the content of the individual TIS tuples: whenever two sets of complementary cardinalities appear within a tuple, they are always complementary sets. This remarkable relation will be explored further below.

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[21]Two apparent exceptions to this statement are the complements of sets 2-6 and 10-4, which are not present. These omissions, however, result from the exclusion of sets 11-1 and 1-1 from the table: 10-6 is held invariant by 11-1, and 2-4 holds 1-1 invariant.

[22]The apparent discrepancy regarding sets of cardinal 10 and 2 is again the result of relationships not shown (see Note 21, above): all six sets of cardinal 10 are invariant subsets of 11-1; and, all of the two-element sets hold 1-1 invariant save 2-6, which is totally invariant. Thus, of the six dyads, only 2-6 is the end of a tuple in the same sense as the other invariant sets.

Subsets of Totally Invariant Sets

It is clear from the definition of the inclusion relation that any totally invariant set holds all of its subsets invariant, including, of course, any of N − 1 cardinality. The discontinuity of tuples at totally invariant sets in Table 2 is thus the result of a convention, introduced by Forte and maintained thus far in the present study, according to which the subsets of these totally invariant sets are disregarded.

A list of totally invariant sets and their N − 1 cardinality subsets is given in Table 3. A comparison with Table 2 shows not only that the sets of cardinal greater than 2 that are ends of tuples in Table 2 are all totally invariant sets, as indicated above, but that the sets of cardinal less than 10 that are beginnings of tuples are all subsets—and thus by definition invariant subsets—of these totally invariant sets. These relationships suggest the construction of another chart based on Table 2, but also incorporating the N − 1 cardinality subsets of totally invariant sets. Such a chart, given in Table 4, synthesizes the content of Tables 2 and 3; it is constructed in the same way as Table 2, with the additional feature that subsets of totally invariant sets are indicated by double lines.[23]

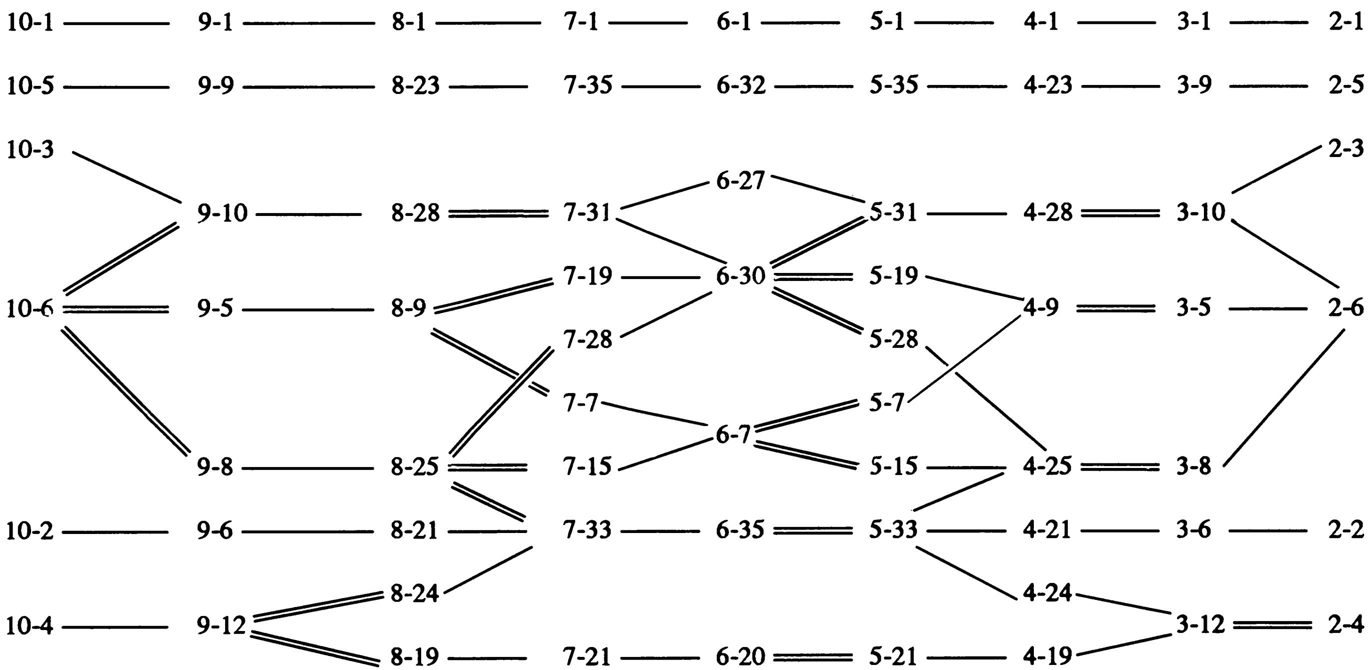
Table 4 displays an exquisitely symmetrical web of interrelationships barely suggested by the structure of Table 2; and yet, the only sets present in Table 4 that do not also appear in Table 2 are 10-6 and 2-4, for reasons described above. The relationships in Table 4 occur in such a way that the table is symmetrical about

[23]The content of Eriksson's Example 2 is similar to that of Table 4, although the relations illustrated are different.

Table 3. Subsets of totally invariant sets.

Pc set	N − 1 subset
3-12	2-4
4-9	3-5
4-25	3-8
4-28	3-10
6-7	5-7
	5-15
6-20	5-21
6-30	5-19
	5-28
	5-31
6-35	5-33
8-9	7-7
	7-19
8-25	7-15
	7-28
	7-33
8-28	7-31
9-12	8-19
	8-24
10-6	9-5
	9-8
	9-10

Table 4. Invariant subsets of cardinal N - 1 for sets of cardinal N, including subsets of totally invariant sets.



an axis consisting of its hexachordal entries; further, symmetrical positions in the table are always occupied by complementary sets. This symmetry suggests the following theorem, a proof of which is offered as an appendix to this paper:

> THEOREM 4: Given two sets A and B, if A holds B invariant under transposition, then the complement of B holds the complement of A invariant under transposition.

In Table 4, the tuples of Table 2 have all become continuous across the entire range of cardinalities. We accordingly define a TIS *string* as follows, to distinguish it from a tuple which does not necessarily span all cardinalities:

> DEFINITION 3: A transpositionally invariant subset (TIS) string is a group of sets whose cardinalities form a continuous sequence 10, 9, 8, ..., 4, 3, 2, with one set for each cardinality represented, in which every set is a transpositionally invariant subset of the next larger set.

Note that this definition, like that of a TIS tuple (Definition 2), invokes the continuous sequence requirement to restrict membership to N − 1 cardinality subsets. All TIS strings in the 12-pc universe are extractable from Table 4.

The Subcomplex Ki

The premise that the N − 1 invariant subset relationship might play a significant role in atonal music is intuitively attractive. Of course, many of the sets participating in this relation are among the most frequently employed in the repertoire; but beyond this straightforward observation, the fact that one set contains another on multiple transpositional levels betokens a degree of relatedness greater than that reflected by the simple inclusion relation. The small number of sets holding this relationship reflects its special nature.

Since the N − 1 TIS relationship, in common with Forte's subcomplex Kh, is essentially a qualified form of the inclusion relation,[24] it is possible to use this relationship as the basis of another subcomplex, which I will call Ki and define as follows:

> DEFINITON 4: Given two sets A and B, B $\in$ Ki($A/\overline{A}$) iff $B/\overline{B}$ and A are members of the same transpositionally invariant subset (TIS) string.

It is necessary to stipulate "B and its complement" in this definition, since otherwise the reciprocal complement relation would not always apply; for example, 4-9 would be an element of Ki(5-31/7-31), since both 4-9 and 7-31 are members of the same TIS string. However, 4-9 is not even a subset of 5-31; it would thus be possible for a Ki subcomplex to contain a set that is not even a subset of the nexus set.[25]

Another way of defining the Ki subcomplex about a set A is as "the set of all TIS strings that include A and its complement." In either case, notice that, because of the symmetry of Table 4 about sets of cardinal 6, the reciprocal complement relation applies "automatically" for Ki subcomplexes about these sets.

Any two sets that are in the relation Ki are also in the relation Kh. This is true because the criteria for membership in Ki

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[24]Forte defines Kh both as a specific relationship between sets and as a group of sets that share this relationship.

[25]This is a property of the set-complex K: for example, 4-18 $\in$ K(5-3) even though 4-18 $\not\subset$ 5-3, because 4-18 $\subset$ 7-3.

includes those for Kh membership, plus the additional stipulation of invariance. Thus, Ki might be thought of as a "sub-subcomplex." In two instances, those of 4-28 and 6-35, Ki and Kh are totally congruent. Ki subcomplexes about sets of cardinal 4, 5, and 6 are displayed in Tables 5-7.

* * *

<u>Analytical Application:</u>

Webern: *Five pieces for String Quartet*, op. 5/4

The analysis subjoined here, while carrying no presumptions of comprehensiveness, demonstrates a potential mode of application of the theory developed in the preceding pages. This analysis integrates invariant-subset relationships into a set-complex view of an entire (albeit brief) composition.

Webern's op. 5/4 is possibly the most frequently analyzed piece in the atonal repertoire. I have chosen to base my set-segmentation of this piece in large part on two previous set-theoretic analyses, those of Beach (1979) and Forte (1980); the results generated here are therefore to be understood less as a function of a particular segmentation strategy than as an extension of the implications for set-complex structure in Beach's and Forte's analyses, and thereby as a demonstration of the potential utility of the Ki subcomplexes.

Both Beach and Forte observe the prominence of sets 5-7 and 4-9 in this piece; Forte also points out multiple occurrences of 3-5. These three sets, each of which is the most frequently occurring set of its cardinality in the piece, are members of the TIS

Table 5. The subcomplexes Ki.
4-element nexus sets

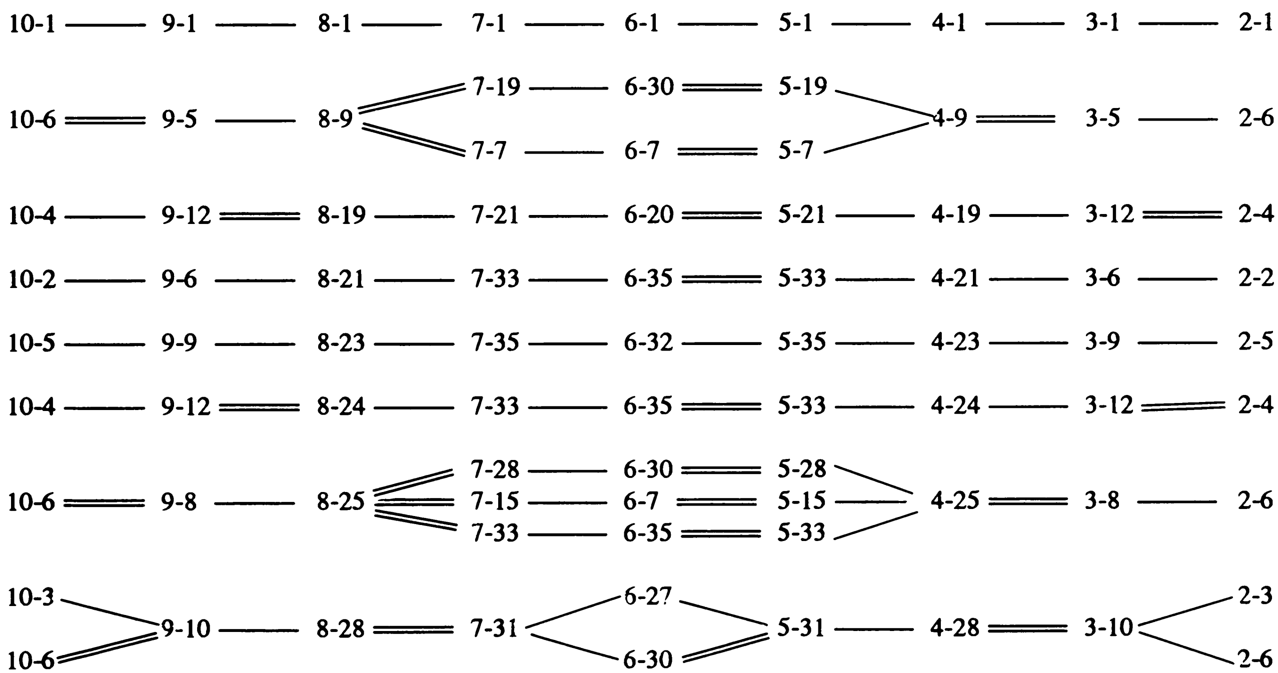

Table 6. The subcomplexes Ki.
5-element nexus sets

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10-1 ——— 9-1 ——— 8-1 ——— 7-1 ——— 6-1 ——— 5-1 ——— 4-1 ——— 3-1 ——— 2-1

10-6 ═══ 9-5 ——— 8-9 ═══ 7-7 ——— 6-7 ═══ 5-7 ——— 4-9 ═══ 3-5 ——— 2-6

10-6 ═══ 9-8 ——— 8-25 ═══ 7-15 ——— 6-7 ═══ 5-15 ——— 4-25 ═══ 3-8 ——— 2-6

10-6 ═══ 9-5 ——— 8-9 ═══ 7-19 ——— 6-30 ═══ 5-19 ——— 4-9 ═══ 3-5 ——— 2-6

10-4 ——— 9-12 ═══ 8-19 ——— 7-21 ——— 6-20 ═══ 5-21 ——— 4-19 ——— 3-12 ═══ 2-4

10-6 ═══ 9-8 ——— 8-25 ═══ 7-28 ——— 6-30 ═══ 5-28 ——— 4-25 ═══ 3-8 ——— 2-6

10-3 \                          / 6-27 \                              / 2-3
       9-10 ——— 8-28 ═══ 7-31                5-31 ——— 4-28 ═══ 3-10
10-6 //                         \ 6-30 //                             \ 2-6

10-6 ═══ 9-8 ——— 8-25 \\                              / 4-25 ═══ 3-8 ——— 2-6
10-2 ——— 9-6 ——— 8-21 ——— 7-33 ——— 6-35 ═══ 5-33 ——— 4-21 ——— 3-6 ——— 2-2
10-4 ——— 9-12 ═══ 8-24 /                              \ 4-24 ——— 3-12 ═══ 2-4

10-5 ——— 9-9 ——— 8-23 ——— 7-35 ——— 6-32 ——— 5-35 ——— 4-23 ——— 3-9 ——— 2-5
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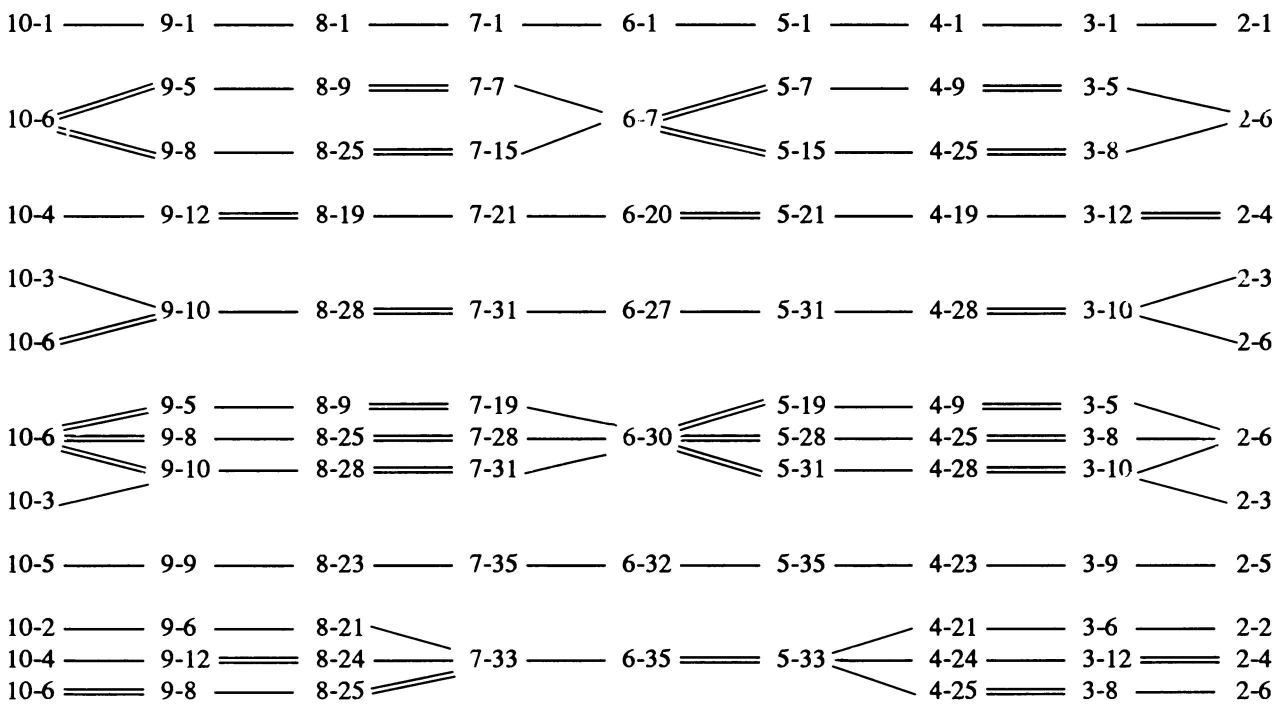
Table 7. The subcomplexes Ki.
6-element nexus sets



string {10-6, 9-5, 8-9, 7-7, 6-7, 5-7, 4-9, 3-5, 2-6}. As shown in the musical example, the complements of these three sets are prominent as well: 9-5, disregarded by Beach, is of particular formal significance, since it represents the total pitch content of each of the outer sections. In fact, the "slow turnover of pitch material" observed by Beach[26] is nowhere as clear as in the overlapping 9-5 formations in mm. 1–6. The 9-5 occurrences in mm. 3–6 and 11–13 also suggest a different interpretation of the unaccompanied linear 7-19 statements in mm. 6, 10, and 13: while Beach considers these passages transitional, their participation in 9-5 formations suggests that they belong more to the passages that precede them than to those that follow, and thus perform a sort of "closing" function. This interpretation is supported by two further observations, in light of the importance of silence and of tempo as devices for formal articulation in Webern's music: first, the 7-19 statements in mm. 6 and 10 begin while other voices are still sounding but are followed by silence; and second, these two statements occur within ritardandos, with the *a tempo* in each case directly following the 7-19 statement.

The set 6-7, which connects the two TIS string-segments {9-5, 8-9, 7-7} and {5-7, 4-9, 3-5}, occurs in mm. 3–4 as the combination of two 4-9 statements in the two violins; the third 4-9, the canonic imitation in the cello, is of course pc-identical to the first. The two remaining sets in this string, 2-6 and 10-6, are also important: 2-6 occurs, for example, as the first two second violin

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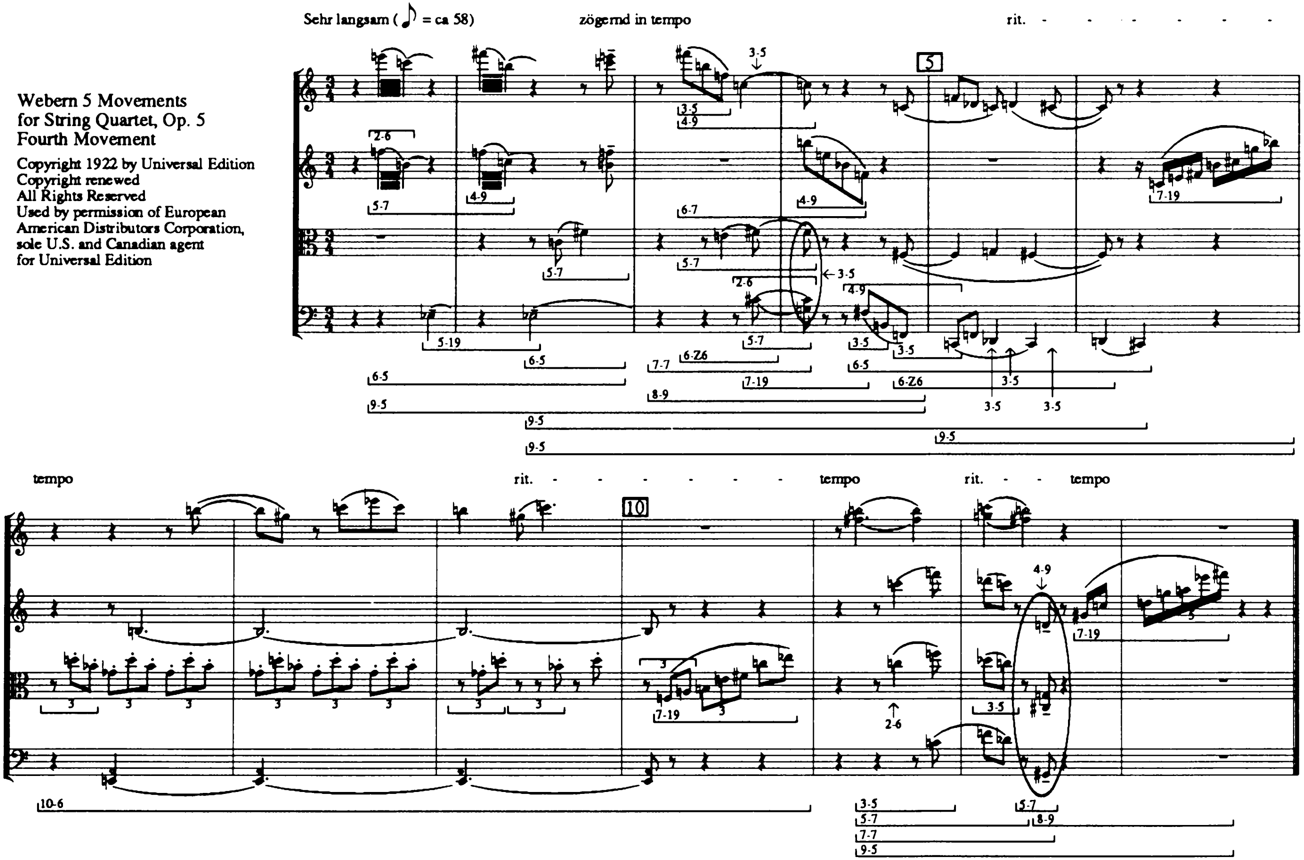
[26]Beach (1979), p. 19.

Webern 5 Movements
for String Quartet, Op. 5
Fourth Movement



pitches in m. 1, the two-note cello figure of mm. 3–4, and the first simultaneity of m. 11; 10-6 represents the total pitch content of the "B" section, mm. 7–10.[27]

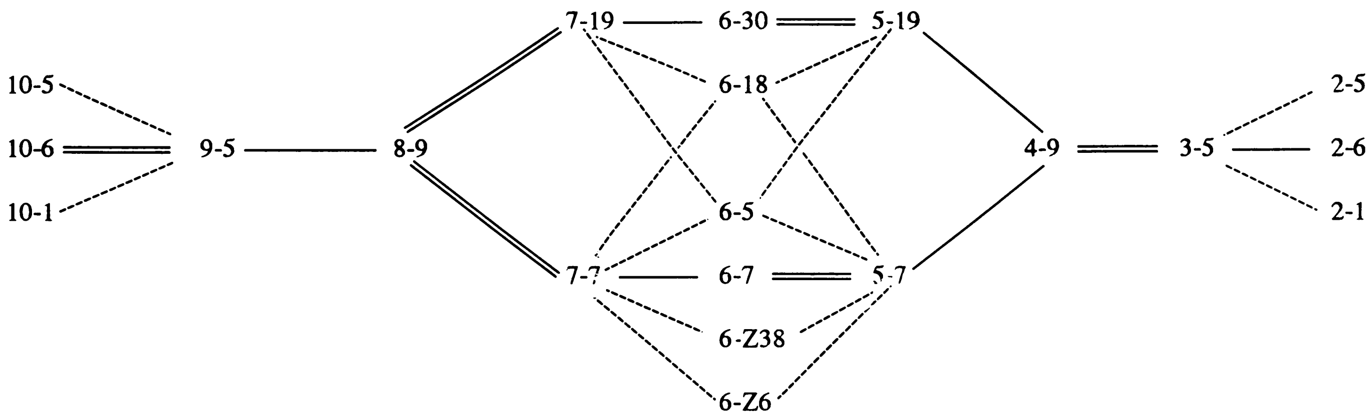
Of the sets that are not members of the string cited above, undoubtedly the most prominent in op. 5/4 is 7-19, a set whose formal role I have already discussed. But this set (along with its complement, 5-19, which occurs in mm. 1–2) is a member of Ki(4-9) as well (see Table 8). Two other, less prominent sets cited by Forte, 6-5 and 6-Z6, while not members of Ki(4-9), are members of Kh(4-9), also shown in Table 8. Thus, the most structurally significant sets in the piece belong to a single TIS string within a Ki subcomplex. Two other important sets are members of the same Ki subcomplex, and two subsidiary sets outside the Ki subcomplex are members of the Kh about the same nexus set. Such thorough exploitation of a single subcomplex in this brief work bespeaks a remarkable consistency of harmonic vocabulary and goes a long way toward confirming our sense of coherence in this enigmatic little piece.

Conclusion

The subcomplex Ki provides a new and unique mechanism for characterizing the relationship held by a relatively small group of sets. While the nonextensibility of this property to all sets in the

[27]The B section has always been the toughest analytical nut to crack in this piece; perhaps the most compelling treatment of this section is that of Burkhart (1980). While the relationship of the B section to the A section remains problematic, this occurrence of 10-6 suggests an intriguing "background" relationship between the two strongly contrasting sections. It is also provocative in light of the fundamental structural role of tritone relationships pointed out by Burkhart.

Table 8. The subcomplexes Kh and Ki(4-9).*



*Kh relations are indicated by dashed lines.

12-pc universe might be regarded as a limitation by those who search for a universal measure of set-class relatedness, this very exclusivity may be its greatest virtue: no previously developed index for measuring set similarity produces a result or value that correlates with Ki membership; that is to say, none of the vector-based similarity indices consistently yields greater values for pairs of sets holding the TIS relation than for pairs holding only the more general property of inclusion. Accordingly, while the Ki subcomplexes do not provide a universal apparatus for measuring set-class relatedness, they do "flag" certain sets which, because of the special properties that characterize the TIS relation, can be of especial importance to composers and analysts. And, as I have tried to show, the abstract relationships held by these sets have a symmetry and elegance all their own.

Appendix

Expressed symbolically, Theorem 4 states the following proposition: If $B \subset A$ and, for some $t \neq 0$, $B \subset T_t(A)$, then $\overline{A} \subset \overline{B}$ and $\overline{A} \subset T_{t'}(\overline{B})$. The following proof will demonstrate that this is true for $t' = -t$. We already know that

1) if $B \subset A$ then $\overline{A} \subset \overline{B}$.

Similarly,

2) if $B \subset T_t(A)$, then $\overline{T_t(A)} \subset \overline{B}$.

Transposing both sides of this expression by -t yields

3) $T_{(-t)}\overline{T_t(A)} \subset T_{(-t)}(\overline{B})$.

Since (as can easily be proved) transposition and complementation are communtative, i.e. $\overline{T_t(A)} = T_t(\overline{A})$, expression 3 can be rewritten as follows:

4) $T_{(-t)}T_t(\overline{A}) \subset T_{(-t)}(\overline{B})$.

Since $T_{t'}T_t(\overline{A}) = T_{(t+t')}(\overline{A})$, this is equivalent to

5) $T_{(t-t)}(\overline{A}) \subset T_{(-t)}(\overline{B})$,

or simply,

6) $\overline{A} \subset T_{(-t)}(\overline{B})$.

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