The Trace, Its Relation to Contour Theory, and an Application to Carter's String Quartet No. 2

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Introduction

Current theories of musical contour have their roots in the atonal theoretical formulations of Robert Morris's *Composition with Pitch Classes*[1] and Michael L. Friedmann's "A Methodology for the Discussion of Contour: Its Application to Schoenberg's Music."[2] Precedents in the ethnomusicological literature abound, including a valuable review and synthesis by Charles R. Adams.[3] Though many of the methodologies in these ethnomusicological studies (for example, reduction of contours to "background" shapes, "flattening" of intervals) overlap with those in the atonal theory literature, the ethnomusicological work typically has the "wider-focus" goal of classifying entire pieces of music within a large body of related samples. With few exceptions, work in the atonal theory field "narrows" the focus to note-to-note relations among a handful of pitches (or other musical parameters), as does the trace model I introduce below.

Usually, a contour is a model or representation of a melody or melodic fragment, that is, pitches in time. Most of the examples below will use relations between pitch and time to illustrate theoretical concepts, but it should be remembered that a generalized contour represents the relationship between any two "orderable" musical parameters. Morris[4] lists examples of ordered parameters, such as "loudness from soft to loud," "timbre-color from dull to bright," and "stereo-space-direction from left to right,"

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[1] Morris 1987, particularly 23-58 and 281-312.

[2] Friedmann 1985.

[3] Adams 1976.

[4] Morris 1987.

following ideas advanced by James Tenney.[5] Parameters other than pitch and time are explored in articles by Elizabeth West Marvin.[6]

A melodic contour represents the "up-and-down-ness" of a melody, showing how pitches in time are related by relative register and relative temporal position, without measuring an actual melodic or time interval. Thus, the excerpts in Example 1 all share the same contour manifested in different ways. Both Morris and Friedmann represent such a melodic contour as an ordered set: a list of the *relative* pitch heights of each note (n distinct pitches ordered from 0, the lowest, to n-1, the highest), written down in their temporal order. The examples in Example 1 are notated as <2 1 0 3> according to this scheme, an ordered set that Morris refers to as a *contour-space segment*, or cseg, and which Friedmann refers to as a *contour class*. To avoid confusion, constructions for which conflicting terminology exists will be referred to using Morris's nomenclature.[7]

A cseg, then, is a basic shape in which both pitch intervals and time intervals are "flattened." Whether or not a cseg models a listener's perception is a question addressed in the music cognition literature.[8] For present purposes, a cseg may be thought of as merely a special representation of a written or heard melody.

In this standard model, two parameters are *ordered* (pitch from lowest to highest, dynamics from softest to loudest, rhythmic points from earliest to latest, etc.), but not *measured.* In other words, a melodic leap of an octave can be represented as equivalent to that of a half-step, the time-interval of a sixteenth-note can be represented identically to that of a whole-note, and so forth, in a given context. The varied time-pitch examples in Example 1 provide a case in point. In its standard form, a cseg can represent, for example, the general "up-and-down-ness" of a melody over time, but purposefully avoids measuring the pitch intervals or the time-distances between successive pitches.

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[5] Tenney 1986.

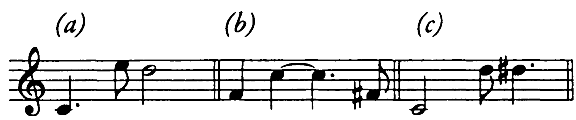
[6] Marvin 1991 and Marvin 1995.

[7] See Friedmann 1987 for a discussion of these conflicts.

[8] Extensive bibliographies of such work are found, for example, in Marvin and Laprade 1987 and Quinn 1997.

Example 1. Three melodic fragments with identical contour representations: (a) Elliott Carter, String Quartet No. 2, III, mm. 314-315, viola part; (b) Robert Schumann, Dichterliebe *(Op. 48), VII, mm. 1-2, vocal part; (c) J. S. Bach, Sinfonia XI, BWV 797, m. 1.*

Example 2. Standard contour similarity measurements would group (a) and (b) as most similar. A trace-based similarity measurement would group (a) and (c) as most similar.



In certain cases, the "flattened" parameters of cseg representations can make for a counterintuitive model for an actual heard melody, particularly when one wishes to judge the similarity of two melodies.[9] For example, any of the Morris- and Friedmann-derived similarity measurements would judge the fragments in Examples 2a and 2b to be identical, and the fragment in Example

[9] Such similarity measurements have been advanced in Friedmann 1985, Marvin and Laprade 1987, Quinn 1997, and Morris 1993. For a model that does take pitch- and time-distances into account, see Juhasz 2000.

2c to be distant from both. A system that takes into account both the shape of the melody and the way it "covers" measured pitch space in measured time might judge Example 2c to be a close neighbor to Example 2a, and Example 2b to be less proximate to them both. Such a system could have valuable applications in the analysis of atonal music, and is a focus of the trace model.

Like a contour, the construction I call a *trace* is a visual and mathematical model for the relationship between two musical parameters. Unlike those in a contour, however, both parameters in a trace may be ordered or measured (or one ordered and one measured), depending on the goals of the analyst looking at a particular passage of music. The various ways of applying ordering or measurement are discussed below.

Imagine creating a visual representation of the melody in Example 1a on the standard x and y axes of elementary algebra. One mapping designates the x-axis (horizontal) as corresponding to time, moving forward from left to right, and the y-axis (vertical) as corresponding to pitch, higher points on the axis representing higher pitches. Once scales for time and pitch have been decided upon, one can then plot four points on the graph, each point corresponding to a given note in Example 1a. This representation (Example 3a) looks something like the standard Western musical notation of the same passage.

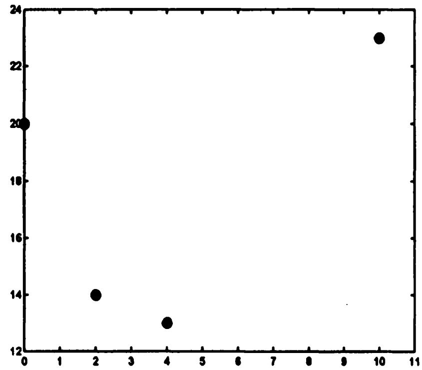
However, the mathematical graph allows for greater flexibility in representation than a standard musical staff. If one wished to convey a sense of each note's full durational value, line segments might be extended from each rhythmic attack point (see Example 3b).[10]

It is the smooth, continuous graph connecting the four points in Example 3c that I call a *trace* of the melody. Like the graphs in Examples 3a and 3b, it represents the four notes of the melody according to their placements in time space and pitch space. Unlike a traditional contour, which represents the notes as discrete entities, this trace shows an imagined, gestural path through pitch-time space. In effect, it de-emphasizes the discrete articulation of the pitches themselves in favor of a continuous flow that connects them.

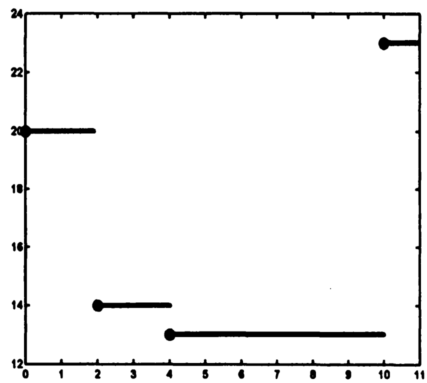
[10] This is akin to the representation proposed in Juhasz 2000.

Example 3. Possible graphic representations of three melodic fragments with identical contour representations.

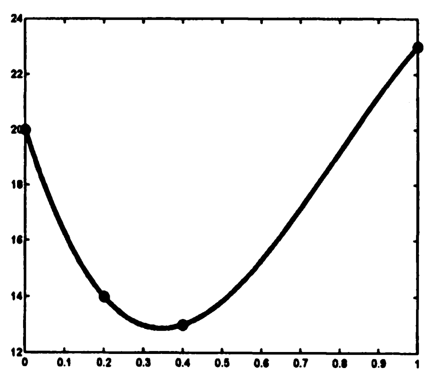
(a)



(b)



(c)



Representing a Melody as a Trace

I will apply the term "trace" to two related ideas: (1) the graph of the relationship between two parameters, as described above, and (2) the mathematical equation that describes such a graph.[11] This section and the next focus primarily on the methods for deriving the mathematical equation from notated music. I will use the Carter passage in Example 1a as an example throughout the section.

Representing a passage of music as a trace involves several steps. First, a decision must be made as to which two musical parameters will be related, and which axes each will occupy. For Example 1a, I will relate time-space (x-axis) to pitch-space (y-axis).

The next step is to determine how each of the two parameters will be represented, that is, how the printed notation will be parsed into numbers. As mentioned above, each parameter may be ordered or measured. By *ordering* a parameter, I mean placing all values in an ordinal ranking according to some set scheme. Methods for doing this include ordering pitches from lowest to highest, rhythmic attacks from earliest to latest, dynamics from softest to loudest, or chord members from lowest to highest. An example in pitch space would be <2 1 0 3> for the example in Example 1a. This procedure follows that used in the standard contour theory of Morris and others.

By *measuring* a parameter, I mean taking into account the sizes of a parameter's intervals, measured by some predetermined scheme.[12] Virtually any measurement units for intervals may be accommodated, typical units being half-steps in pitch-space, quarter notes (or fractions thereof) in time-space, or standard dynamic markings in loudness-space. Fundamentally different from ordering, the measured approach mimics more closely standard Western notation of melodic material.

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[11] Note that this usage is unrelated to the use of the term "trace" in linear algebra. See Strang 1986.

[12] Here I use the term *interval* broadly, encompassing not only the distance between two pitches, but also the "time-distance" between two rhythmic points, the "loudness-distance" between two dynamic levels, and so forth. See Lewin 1987.

Each approach (ordered and measured) models a musical passage. The ordered approach posits an underlying, essential "shape" to the relationship between the two parameters, no matter how widely or narrowly the values may vary. The measured approach sacrifices the ideal background "shape," and models a sense of depth or shallowness in the parametric relationships. One's goals in analyzing the specific passage or passages of music will dictate which approach is best suited for an analysis. In some cases, a hybrid approach (in which one parameter is measured and another ordered) may be the most fruitful.

In creating a trace for the musical passage in Example 1a, I will employ the measured model for both parameters. Given Carter's pitch language, an appropriate measurement scale for pitch-intervals is half-step increments.[13] Given his rhythmic language, quarter notes (or fractions thereof) serve as a scale for time intervals. The convention I use here is to graph rhythm along the x-axis, and pitch along the y-axis of the graph. As one goal of this article is to offer an analysis of this Carter movement, I will generally tailor traces and other trace-related constructions to forms that I think reveal the most about the movement, though alternative constructions may be alluded to in passing.

Using these measurement schemes, the passage in Example 1a can be represented as a set of (x,y) ordered pairs, where each x-value corresponds to the location of a note in time-space, and each associated y-value corresponds to the location of the same note in pitch-space. Suppose that our time-line begins with 0 and increases by 1 for each quarter note that passes after the beginning of m. 314. Further suppose that each pitch is represented by its distance from C4 in half-steps. Then a list of (x,y) pairs would look like the set in Table 1.

Table 1.

x	0	2/3	1 1/3	3 1/3
y	20	14	13	23

[13] In other cases, scale degrees might be more appropriate, or octave-spans, or even frequency in Hz.

Note that the durational values of the printed notes are not taken into consideration; merely the time of the onset of each note is indicated. In particular, the final note of the four (B5) could be of any rhythmic length, yet the diagram of (x,y) pairs above would remain unchanged.

Pitch-intervals, of course, need not be measured from a fixed reference pitch, but may be defined relative only to each other. In this case, the starting pitch could be set to zero (much the way the onset time of the first note was set to zero), and the other pitches measured relative to it. Applying this method to the phrase in Example 1a, the set of (x,y) pairs in Table 2 would result.

Table 2.

x	0	2/3	1 1/3	3 1/3
y	0	-6	-7	3

I will term any set of (x,y) pairs that, like Table 2, contains the pair (0,0) a *self-aligned* set. Any self-aligned set of pairs will have a point of reference, within the set itself, that is set to (0,0) for convenience. Sets of pairs like those in Table 1, above, for which at least one variable is measured from a referential zero-point outside the set itself, I will call *referent-aligned.*

To compare sets of pairs (and thus their traces) more intuitively, one or both of the parameters might also be *normalized,* or scaled down to a standard, convenient size. For Table 2, one could merely divide the x-part of each pair by the largest value of x in the set, as shown in Example 4.

The numbers on the right-hand sides of the equations in Example 4 can be substituted for the original x-values, and are proportionately spaced just as those in Table 2. The difference is that the "new" values range from 0 to 1 instead of from 0 to 3 $^{1}/_{3}$. This process yields another form of the (x,y) pairs, as shown in Table 3.

One could normalize the y-variable as well, narrowing the values between -1 and 1, for instance, but Table 3 already illustrates a useful form for representing time-pitch relationships. The format in Table 3 will be the standard in representing melodic fragments in this paper: both variables are measured, the x-variable (duration) is normalized, and the set is self-aligned.

Once an appropriate form for the (x,y) pairs has been chosen, it is relatively simple to create the corresponding trace (the curve constructed to connect all the points, as seen in Example 3c), especially with the aid of a computer.[14] The following procedure derives the mathematical equation for a curve connecting all of the (x,y) points on a graph.

Example 4. Normalization of the values in Table 2.

$$0 \div \left(3\frac{1}{3}\right) = 0.00$$

$$\frac{2}{3} \div \left(3\frac{1}{3}\right) = 0.20$$

$$\left(1\frac{1}{3}\right) \div \left(3\frac{1}{3}\right) = 0.40$$

$$\left(3\frac{1}{3}\right) \div \left(3\frac{1}{3}\right) = 1.00$$

Table 3.

x	0.00	0.20	0.40	1.00
y	0	-6	-7	3

[14] All graphs and computations in this article were performed using MATLAB.

Deriving the Trace from Ordered Pairs

Recall that in standard two-dimensional Euclidean space, two discrete points determine a line; that is, there exists an unique line that intersects the two points. Remember, too, that a line can be represented by the equation $y = a_1x + a_0$, a first-degree polynomial equation. Given the coordinates of any two points, one can derive the constants a_1 and a_0, thus finding the equation for the line that passes through them. If we try to find a second-degree polynomial equation $y = a_2x^2 + a_1x + a_0$ that passes through the same two points, we will discover that there is more than one equation which does so. For a set of two distinct points, only a first-degree polynomial has this property of uniqueness. Generally, n points can be uniquely described by a polynomial of degree $n-1$. As an example, I will determine the equation for Table 3, the set of ordered pairs we constructed for the Carter string quartet example.[15]

Since there are four distinct (x,y) points, the equation will be of the form $y = a_3x^3 + a_2x^2 + a_1x + a_0$; in other words, the goal is to find a_0, a_1, a_2, and a_3, such that all the (x,y) pairs will satisfy the equation. The resulting set of simultaneous linear equations can be represented by the matrix equation in Example 5a. This set of equations can be solved by a variety of methods, yielding the solution in Example 5b.[16]

The equation intersecting all four points is $y = -35.42x^3 + 83.75x^2 - 45.33x$. The right-hand side of this equation is what I will call the *trace polynomial* for the set of pairs in Table 3. I will refer to the solution vector in Example 5b as the *trace vector* for the set of pairs in Table 3, essentially a shorthand version of the trace polynomial. The graph of this polynomial equation is shown in Example 6. Because the set of ordered pairs was defined so that

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[15] Several procedures for fitting a curve to data points exist. I have chosen to concentrate on polynomial curve-fitting because it makes the subsequent comparison of two traces computationally simple, as shown below. Other methods, such as spline curve-fitting, are certainly possible, and indeed have some advantages in reducing error, but create computational complexity when comparing traces.

[16] In Example 5b, all numbers have been rounded to two decimal places.

Example 5. (a) Matrix equation utilizing values in Table 3; (b) Solution of this equation.

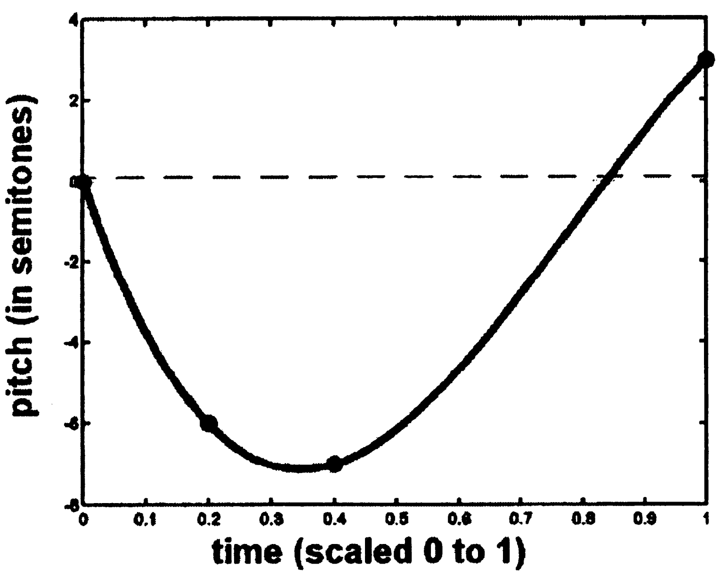
(a)

$$\begin{bmatrix} 0 \\ -6 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0^3 & 0^2 & 0 & 1 \\ (0.2)^3 & (0.2)^2 & 0.2 & 1 \\ (0.4)^3 & (0.4)^2 & 0.4 & 1 \\ 1^3 & 1^2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} -35.42 \\ 83.75 \\ -45.33 \\ 0.00 \end{bmatrix}$$

Example 6. A trace representation of the melody in Example 1a.



(0,0) was included, the resultant graph will always pass through the origin, and the last entry in the trace vector (corresponding to the coefficient a_0) will always be zero. Deriving the trace allows one to represent the notated music mathematically, to describe it compactly with a vector of numbers, and to represent it as a graph.

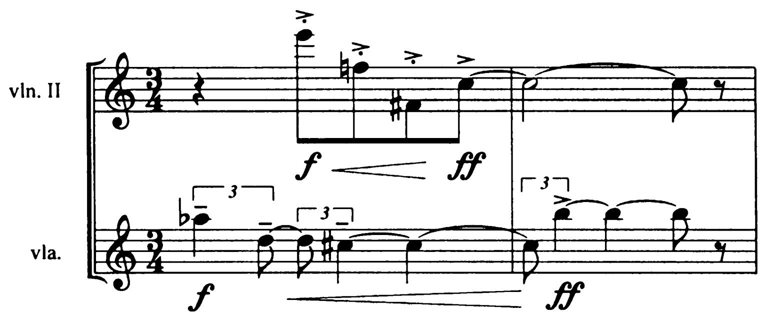
Comparing Two Melodic Traces; Trace Transformations

Here I posit a transformational approach to studying relationships between traces. Certain features of a transformation that carries one trace to another can be quantified and categorized as befits analysis of the passage. The most salient feature of this method is that any trace can be transformed into any other trace, no matter the cardinality of the vectors (that is, the number of ordered pairs) and no matter how dissimilar their trace graphs may appear. For example, finding the function that transforms the trace of a seven-note melody into the trace of a three-note melody presents no special problems.

The viola passage from mm. 314-315 of the 3rd movement of Elliott Carter's String Quartet No. 2 shown in Example 1a is part of a stretto-like texture. In the same two measures, the second violin has a similar (though not identical) contour. Both instruments' parts are shown in Example 7.

Frequently, in analyzing a passage of music such as the one in Example 7, it is useful to have a means of comparing the melodic motion of two interconnected parts. Similarity measurements have been proposed for and successfully applied to many musical structures, including melodic contours (a few similarity measurement systems for contours are alluded to above). Here I will describe a Lewinian transformational model that maps one trace to the other, and I will derive a similarity measurement from features of the model.

Example 7. Carter, String Quartet No. 2, III, mm. 314-15, violin 2 and viola parts.



Let $\bar{a}$ and $\bar{b}$ be two trace vectors, where $\bar{a} = [a_n\ a_{n-1}\ \ldots\ a_0]$ and $\bar{b} = [b_m\ b_{m-1}\ \ldots\ b_0]$, derived from two sets of (x,y) pairs, each of which has normalized x values. The number of elements in $\bar{a}$ and $\bar{b}$ may be the same or may differ. Call the related trace equations a(x) and b(x), where $a(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, and where $b(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0$. Then the *distortion function* $D(\bar{a}, \bar{b})$ that sends $\bar{a}$ to $\bar{b}$ is defined by the equation in Example 8.

Example 8. The distortion function.

$$D(\bar{a}, \bar{b}) = b(x) - a(x)$$

The distortion function is precisely that transformation which turns a(x) into b(x). Further, we can define the *distortion vector* $\bar{D}(\bar{a}, \bar{b})$ as the vector representation of $D(\bar{a}, \bar{b})$ (its vector of coefficients). As mentioned above, deriving the distortion function does not depend on the cardinality of the trace vectors matching—any trace may be transformed into any other trace.

To calculate the distortion function that relates the second violin and viola traces from Example 7, one must first transform the musical notation into two sets of ordered pairs following the format decided upon above. The two sets of pairs are shown in Table 4.

Table 4.

violin II	x	0.00	0.33	0.67	1.00
	y	0	-11	-22	-16
viola	x	0.00	0.20	0.40	1.00
	y	0	-6	-7	3

Note that the x values for violin II are evenly spaced between 0 and 1, because the onsets of the four notes are equally spaced at an eighth note's duration. Note also that each instrument's set of ordered pairs represents its opening pitch as (0,0). This analysis attempts to relate the melodic motion of the two fragments, not as they overlap in time-space or pitch-space, but as they imitate each other, each unfolding in its own time-scale and tessitura. This way of comparing two melodies has a parallel in the analysis of fugue, where one might compare a fugue subject with its tonal answer, or with a version in rhythmic diminution. Here, the distortion function describes what kind of action or operation would distort the shape of one trace graph into the shape of the other.

Once the two sets have been determined, the trace vectors can then be derived for each melodic fragment, yielding the trace vectors for the violin II and viola lines shown in Examples 9a and 9b, respectively. All that remains is to use these values in the vectors to calculate the distortion function, as defined in Example 8. The distortion vector that transforms the violin II fragment into the viola fragment is $\overline{D}(\overline{vlnII}, \overline{vla})$, a vector of the arithmetic differences, the values of which are shown in Example 10. The reverse transformation, $\overline{D}(\overline{vla}, \overline{vlnII})$, can be obtained by placing a minus sign in front of $\overline{D}(\overline{vlnII}, \overline{vla})$. Example 11a is a graph of these two traces and the area between them. The distortion function, as the difference between two traces, is also a trace (though one that is more "abstract" than a melodic trace, as it is

Example 9. Trace vectors of (a) the violin II line, and (b) the viola line from Example 7.

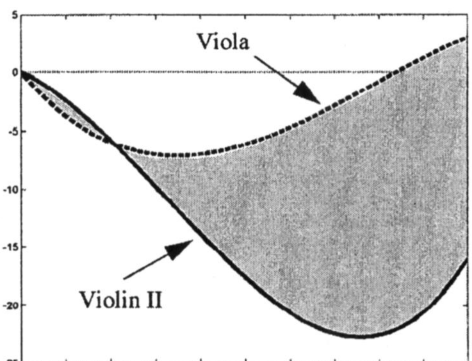
$$(a)\ \begin{bmatrix} 74.0 \\ -72.5 \\ -17.5 \\ 0 \end{bmatrix} \qquad (b)\ \begin{bmatrix} -35.4 \\ 83.8 \\ -45.3 \\ 0 \end{bmatrix}$$

Example 10. The distortion vector that transforms the violin II line into the viola line.

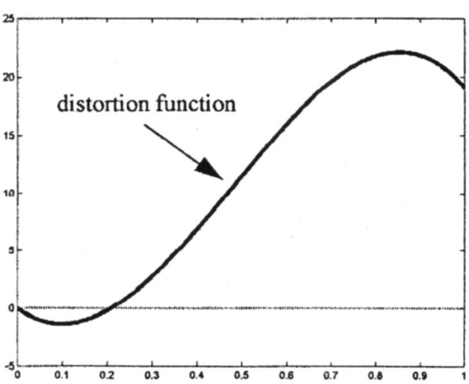
$$\begin{bmatrix} -109.4 \\ 156.3 \\ -27.8 \\ 0 \end{bmatrix}$$

Example 11. (a) Two traces based on the passage in Example 7, and (b) the distortion function that transforms one into the other.

(a)



(b)



not the representation of a sounding entity), and thus it can be graphed. Example 11b is the graph of $\overline{D}(\overline{\text{vla}}, \overline{\text{vlnII}})$.

Note that the distortion graph attains values of zero (that is, it crosses over the x-axis) at precisely the points where the two instrumental trace graphs cross, and the region where the distortion graph is close to zero (the left-hand side of the graph) is precisely where the two instrumental graphs are very close together. In these regions (corresponding to spans of time in the sounding music), the traces do not differ by much, and it takes little "work" to transform one into the other. The right-hand side of the distortion graph, by contrast, attains large positive y values, representing the upward "push" necessary to send the second violin's graph up to the viola's graph in this region.[17]

The information in the distortion function can be interpreted in many ways. One quantity which can be derived from it I call the *integral distance* measurement, which is a metric for the "distance" between two traces; it is well-suited to the task of comparing melodic traces.

Example 12. Formula for integral distance.

$$d_I(\overline{a}, \overline{b}) = \int_0^1 | b(x) - a(x) | \, dx$$

The integral distance between any two trace vectors $\overline{a}$ and $\overline{b}$ is defined by the equation in Example 12, where a(x) and b(x) are the trace equations corresponding to the trace vectors. The upper and lower limits of the integral can be altered to match the domain of x

[17] Here I rely on an extension of the "displacement" concept that is usually associated with voice leading transformation (see Lewin 1998 and Straus 2003) into the realm of melodic transformation. Assume two arbitrary melodies M1 and M2, which have been shifted in pitch-space so that their first pitches align, and which have been rhythmically normalized. If transforming M1 into M2 involves the alteration of a single pitch by a half-step, or a single attack point by a small rhythmic value, I would categorize the transformational "work" as slight. By contrast, if several large pitch or rhythmic adjustments must be made, then more "work" must be exerted to transform M1 into M2.

values desired; for sets of pairs that have been normalized, 0 and 1 (as shown in Example 12) are the appropriate choices.

The integral distance between the violin II and viola traces from the Carter example is exactly the area of the space bounded by the two trace graphs, shaded in Example 11a. This measurement is symmetrical; that is, $d_I(\overline{a}, \overline{b}) = d_I(\overline{b}, \overline{a})$, and the result is meant to describe the total amount of "work" that must be done to transform one trace into the other. The integral may be calculated by hand, or software approximations may be used, to obtain the result $d_I(\overline{\mathrm{vlnII}}, \overline{\mathrm{vla}}) \cong 11$. The result 11 is contextual, taking on meaning only when compared to other integral distance measurements that an analyst deems related. In cases wherein a large number of traces are compared to each other, a clustering method might be used to contextualize the measurements. Unlike various similarity measurements created for contours (see footnote 9), the integral distance does not measure merely the similarity of the traces' overall shapes, but also takes into account their relative spatial proximities. In musical terms, the integral distance measurement compares the way two melodies move through their local pitch spaces (as defined by their opening pitches) over time.

The traces compared in this section were both derived from what I have called "self- aligned" sets: sets in which the coordinate (0,0) is set to correspond with the opening pitch. In other applications, one might instead wish to vertically shift the trace graphs so that the x-axis corresponds to the average value each trace function attains. Such shifting would not privilege the opening pitch as much as the method given above. However, given the stretto-like passages in the Carter movement to be analyzed, wherein imitation is expected to some degree, I hear the incipient pitch taking on an added importance, serving as an anchor for the subsequent pitches in the melodic fragment. Therefore, weighting the relative importance of opening pitches through this trace representation seems fitting in context, and makes little mathematical difference in the application to this movement's examples in any case.

Trace Approximations

One limitation of the trace model stems from the use of polynomials for interpolating the curve between given ordered pairs. As the number of data points increases, the graph of the polynomial tends to fluctuate more wildly, reaching y values far above and far below the general range of the original data points. The curve becomes less and less representative of the pitch space outlined by the pitches. Example 13 shows two melodic fragments that ideally would have very similar trace graphs. The mathematical interpretation, however, fails to follow musical instinct. Note the "wavier" appearance of the trace graph for Example 13b (shown with a dotted line), and especially note that it dips below the actual range of the printed music, particularly at the extreme left-hand and extreme right-hand ends of the graph. Fortunately, a relatively simple corrective exists.

The demonstrated procedure for finding a trace vector from a given set of ordered pairs constructs a vector with cardinality equal to the number of given pairs. If there are n ordered pairs, the trace polynomial has degree n-1, and the vector has n entries. For sets of pairs that give rise to counterintuitive graphs, however, approximating the data with a lower-degree polynomial (shorter vector) may result in a curve that follows musical sensibilities more closely.

A simple and widely-used algorithm for curve fitting is the least-squares method.[18] By using this approximation, one finds a graph which does not "swing" as dramatically up and down, and which passes as closely as possible to the given points. Most of the time, the curve will not pass through the points corresponding to the originally-given pairs, but if the polynomial degree is chosen wisely, the amount of error is negligible.

Example 14 reproduces the trace graph of Example 13b. It was noted above that the extreme left and extreme right sides of the graph have an especially "wavy" appearance. This waviness can be softened by using a least-squares approximation. The originally-derived trace vector for the melodic fragment is shown in Example 15a. An apt least-squares approximation yields a vector with only

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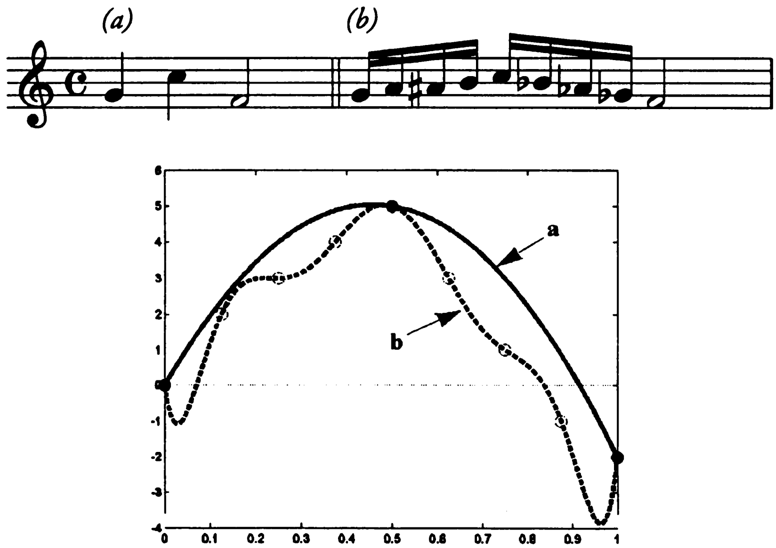
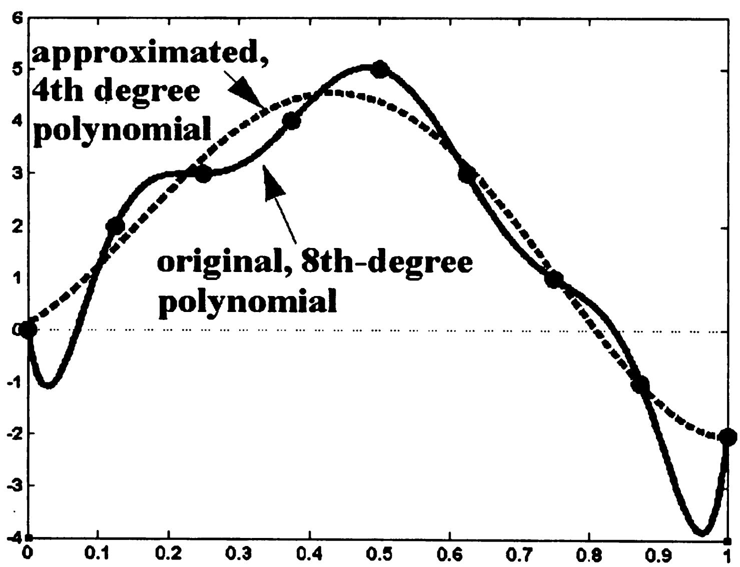
[18] For details about least-squares approximation, see Strang 1986.

Example 13. Two similar melodic passages and their trace graphs.



Example 14. Graphs of originally-derived trace and approximated trace. Circles represent the points derived from the original set of ordered pairs.



five entries (a polynomial of degree 4), is shown in Example 15b. Though the vector representations appear very dissimilar, the graphs are actually quite close, as shown in Example 14. The circles in the figure indicate the points derived from the set of ordered pairs, and the fourth-degree polynomial plot follows them quite closely, though it does not actually intersect a single one of them perfectly. Notice that the final entry in the shorter (approximation) vector is not equal to zero, indicating that the associated graph does not pass precisely through the origin. Despite the fact that the approximation trace fails to intersect the points corresponding to the pitches of the printed music, the visual impression mimics the melodic shape more accurately than does the original, unapproximated trace.[19]

Example 15. (a) The original trace vector for Example 13b; (b) a least-squares approximation of Example 15a.

$$(a)\ \begin{bmatrix} 24970 \\ -99450 \\ 161290 \\ -136620 \\ 64410 \\ -16640 \\ 2120 \\ -90 \\ 0 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 71.61 \\ -131.15 \\ 50.15 \\ 7.25 \\ 0.13 \end{bmatrix}$$

[19] As mentioned above, future work may explore the use of cubic splines for curve-fitting, which would alleviate the problem of "waviness," but the least-squares method is simple and effective in the majority of cases.

Application to Carter's String Quartet No. 2

In Elliott Carter's String Quartet No. 2, each of the four instruments has its own unique character, individuated by the melodic intervals and rhythms that predominate.[20] The third movement, "Andante espressivo," attempts to reconcile the different instruments' "personalities" in a contrapuntal texture that is mostly imitative. Much of this movement is built out of short melodic fragments, all with similar contours, played in different rhythms and staggered through two, three, or four of the instruments, somewhat in the manner of a stretto. This analysis uses traces of the melodic fragments to model and comment upon the shifting imitative relationships among the instrumental parts. Though the stretti and their changing relationships form only one kind of activity in the movement, it is a persistent and important one, and one not examined deeply in the existing literature.[21]

The first stretto of the movement occurs between the viola and the cello in their opening measures (mm. 286-288, see Example 16a). For the most part, I will confine this analysis to passages in which at least three instruments have imitative lines, but as this passage opens the movement, it bears special consideration. The two instrumental parts here have identical always-rising contours, csegs <0 1 2 3 4> in the system of Morris and Friedmann. It is possible to characterize their relationship more exactly by comparing traces of the two fragments.

Table 5.

va	x	0	.33	.56	.78	1
	y	0	5	10	11	16
vc	x	0	.32	.59	.81	1
	y	0	10	18	25	31

[20] See the composer's foreward to the score, Carter 1961.

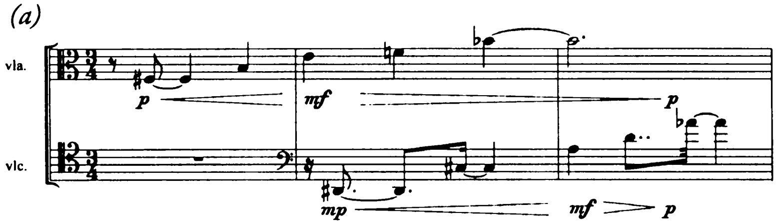
[21] Portions of the quartet are analyzed in Gass 1981, Bernard 1993, Koivisto 1996, and Schiff 1998, but none addresses the third movement.

Using the representation decided upon earlier (measured and normalized x values, measured y values, and self-aligned contours), the sets of ordered pairs in Table 5 are derived, with the x-axis corresponding to time and the y-axis corresponding to pitch. Note that both sets have been rhythmically normalized between 0 and 1, and that each is "transposed" in pitch and "shifted" in time so that they share a common starting point of (0,0). The trace vectors for these two passages, and the integral distance between them, are calculated according to the methods introduced above, yielding an integral distance of 8.2. Example 16b contains the trace graphs of each. In a relatively simple passage such as this one, the trace analysis merely confirms what can be observed in the printed music: both melodic fragments continually rise, the cello's line covering a larger span than the viola does. In analyses of more complex passages, however, trace representations can serve to highlight relationships that are buried in the notated music.

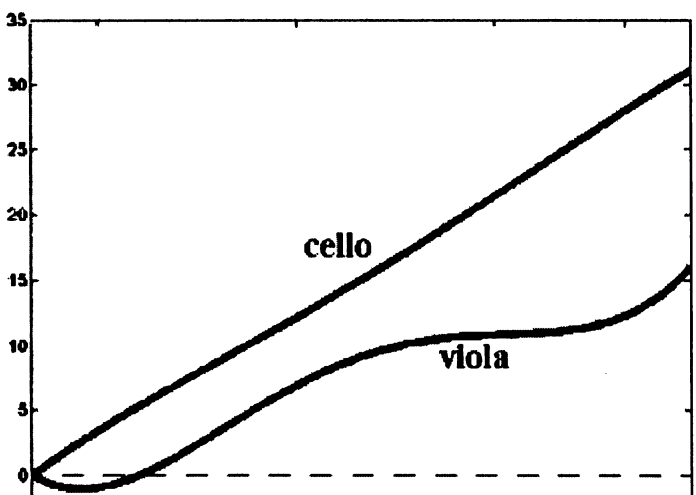
The few measures following the passage in Example 16a contain non-imitative material and stretti consisting of only two notes. The next passage to which trace analysis brings unique insight occurs in mm. 294-296, as shown in Example 17a. The three-note fragments have the same Morris/Friedmann contour <0 2 1>, but differ in the range of pitch space that they fill and, to a lesser extent, in the relative rhythmic durations of the constituent notes. As can be seen in Example 17b, the viola, second violin, and cello all occupy a similar span of pitch space and move through it in a similar way. The first violin, on the other hand, has a much shallower range. This difference is reflected in the integral distances among the four instrumental parts. The viola, second violin, and cello traces have integral distances from each other ranging from 1.1 to 3.7, while the integral distance from the first violin's trace to those of the other members of the ensemble never is smaller than 5.2. The *similarity diagram* in Example 17c attempts to condense all the integral distance information from this passage into a compact form. The lengths of the line segments connecting the instrument names are not in the exact mathematical ratios of the numbers associated with them; instead, they are an approximate

Example 16. Carter, String Quartet No. 2, III, mm. 286-88: (a) musical example, and (b) trace graphs.

(a)



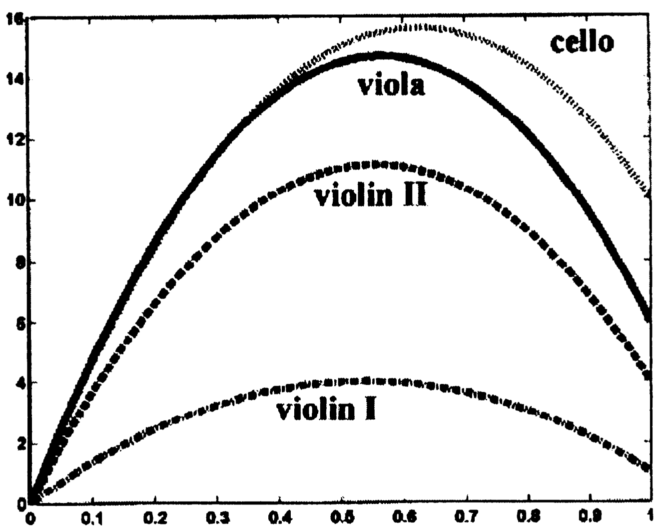
(b)



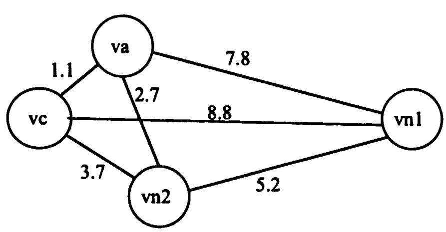
Example 17. Carter, String Quartet No. 2, III, mm. 294-96: (a) musical example, (b) trace graphs, and (c) similarity diagram derived from integral distance calculations.

(a)

(b)



(c)



indication of the relative distances.[22] In particular, the first violin part is depicted as the most remote from the others, which is characteristic in this movement.

Using the data in the similarity diagram, an average integral distance for the passage can be calculated (simply an average of all the integral distances among all the traces). This statistic roughly indicates the degree of similarity among the parts involved. The average integral distance of 4.9 for this passage will be compared to that of other passages below.

Example 18 contains similarity diagrams for nineteen stretti in this movement. The abbreviated instrument names in the left portion of the figure are sometimes accompanied by an asterisk, indicating that the trace vectors associated with them were calculated using a least-squares approximation in order to represent the printed music more closely. Before discussing the set of nineteen passages as a whole, I will point out features of a few of them.

One such stretto begins in m. 300 with the cello's F♯2 (see Example 19a). The cello's fragment is six notes long, ending with the E3 in m. 303. Each of the other instruments has a similar pattern of soft legato notes followed by loud staccato notes, but the total number of notes is not held constant from instrument to instrument. The first violin, cello, and viola each have six notes while the second violin has five. Trace representation of this passage reveals that the viola and second violin have the smallest integral distance from each other (that is, they are the most similar in the way they move through their local pitch spaces over time), despite having different numbers of notes and different contours. The trace graphs in Example 19b illustrate this similarity; the graphs of the viola and second violin traces are both shallow, occupying the same general space.

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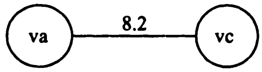
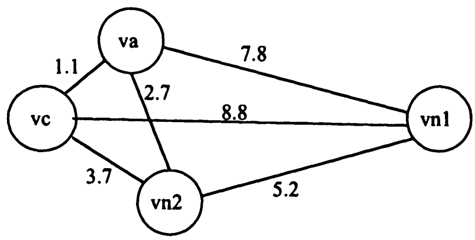
[22] To represent the distances accurately would require a three-dimensional graph in most cases, a visually complex figure at odds with the goal of creating a simple and quick graphical representation. However, in cases wherein more than four or so trace graphs are compared, one may wish to employ multidimensional scaling, as in Samplanski 2004.

Example 18. Similarity diagrams for nineteen stretto passages in the third movement of Carter, String Quartet No. 2.

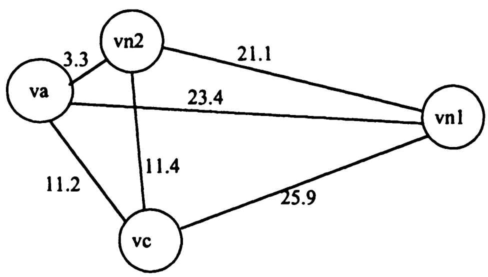
(a) va mm. 286-287
vc mm. 287-288
avg d_I = 8.2



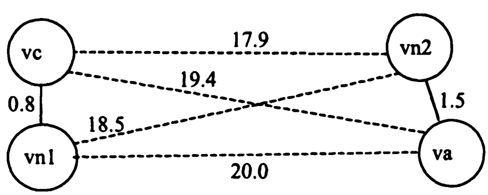
(b) va mm. 294-295
vn2 m. 295
vc mm. 295-296
vn1 m. 296
avg d_I = 4.9



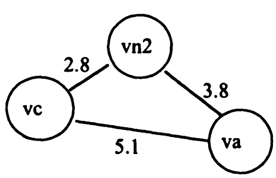
(c) va* mm. 300-302
vc mm. 300-303
vn2* mm. 301-303
vn1* mm. 300-303
avg d_I = 16.1



(d) va mm. 310-312
vn2 mm. 310-311
vc mm. 310-315
vn1 mm. 311-315
avg d_I = 13.0

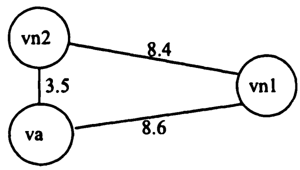


(e) va mm. 316-317
vc mm. 316-318
vn2 mm. 316-318
avg d_I = 3.9

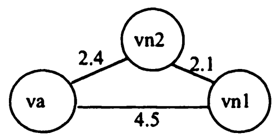


Example 18. Similarity diagrams for nineteen stretto passages in the third movement of Carter, String Quartet No. 2, continued.

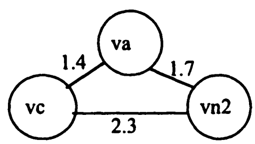
(f) va* mm. 318-319
vn1 mm. 318-319
vn2 mm. 318-319
avg $d_I = 6.8$



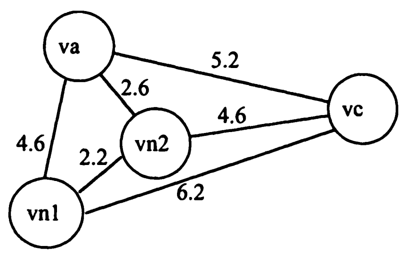
(g) va mm. 322-325
vn2 mm. 324-325
vn1* mm. 324-325
avg $d_I = 3.0$



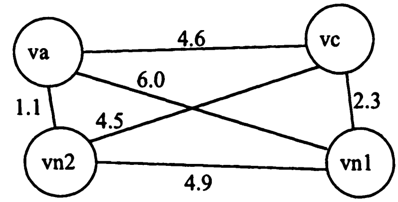
(h) va mm. 327-328
vc mm. 327-329
vn2 mm. 327-328
avg $d_I = 1.8$



(i) va* mm. 329-332
vn2* mm. 329-333
vc* mm. 329-333
vn1* mm. 330-334
avg $d_I = 4.2$

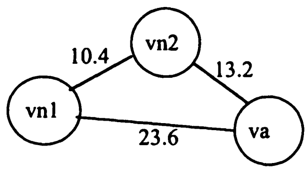


(j) va mm. 333-334
vc mm. 334-336
vn2 mm. 334-336
vn1 mm. 335-336
avg $d_I = 3.9$

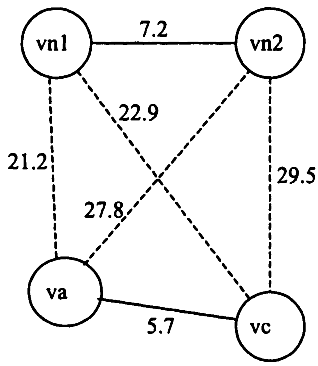


Example 18. Similarity diagrams for nineteen stretto passages in the third movement of Carter, String Quartet No. 2, continued.

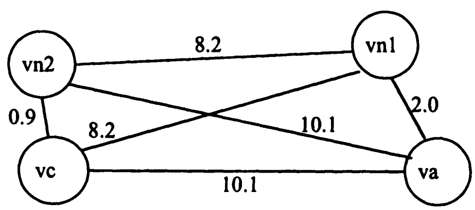
(k) vn2 mm. 337-338
vn1 mm. 338
vn2 mm. 337-338
avg d_I = 15.7



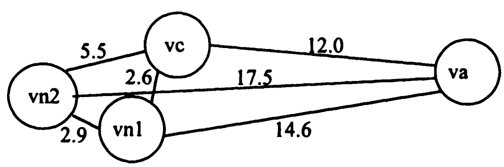
(l) va mm. 338-340
vc mm. 339-340
vn2 mm. 339-340
vn1* mm. 339-340
avg d_I = 19.1



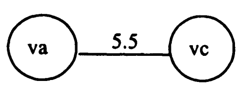
(m) va mm. 340-342
vn1* mm. 340-346
vc mm. 341-346
vn2* mm. 341-346
avg d_I = 6.6



(n) va mm. 346-347
vc mm. 347-349
vn2 mm. 347-350
vn1 mm. 347-351
avg d_I = 9.2

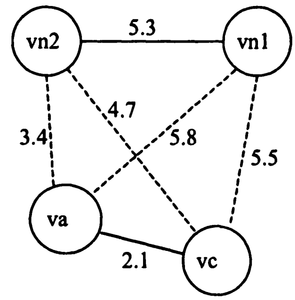


(o) va mm. 350-352
vc mm. 351-355
avg d_I = 3.9

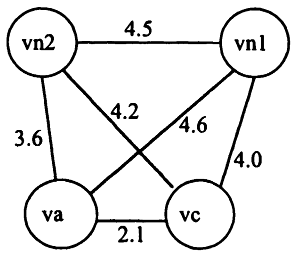


Example 18. Similarity diagrams for nineteen stretto passages in the third movement of Carter, String Quartet No. 2, continued.

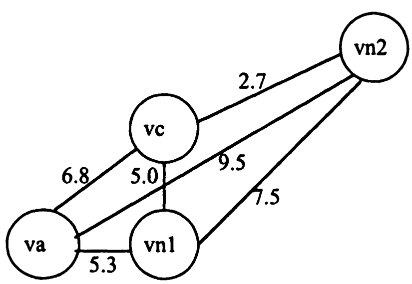
(p) vn2 mm. 355-358
vn1* mm. 355-358
va mm. 356-358
vc mm. 356-358
avg d_I = 4.0



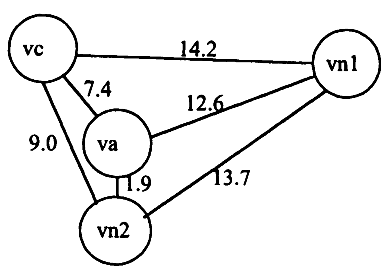
(q) va mm. 358-359
vn2 mm. 360-361
vn1* mm. 361-362
vc m. 363
avg d_I = 3.8



(r) va mm. 363-366
vn1 mm. 363-367
vc mm. 364-367
vn2 mm. 364-366
avg d_I = 6.1

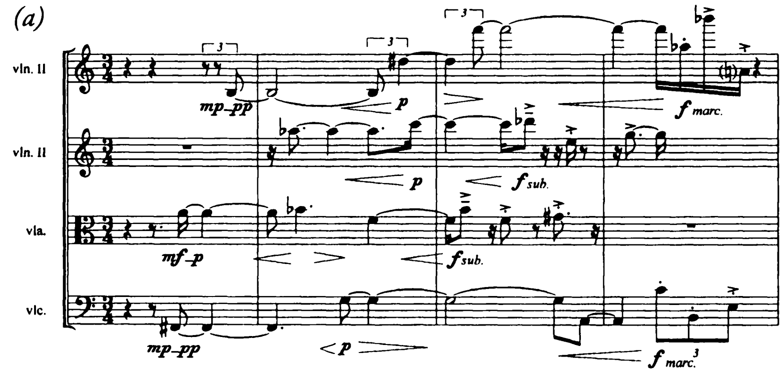


(s) va* mm. 368-372
vc mm. 369-373
vn2* mm. 369-373
vn1* mm. 374-377
avg d_I = 9.8

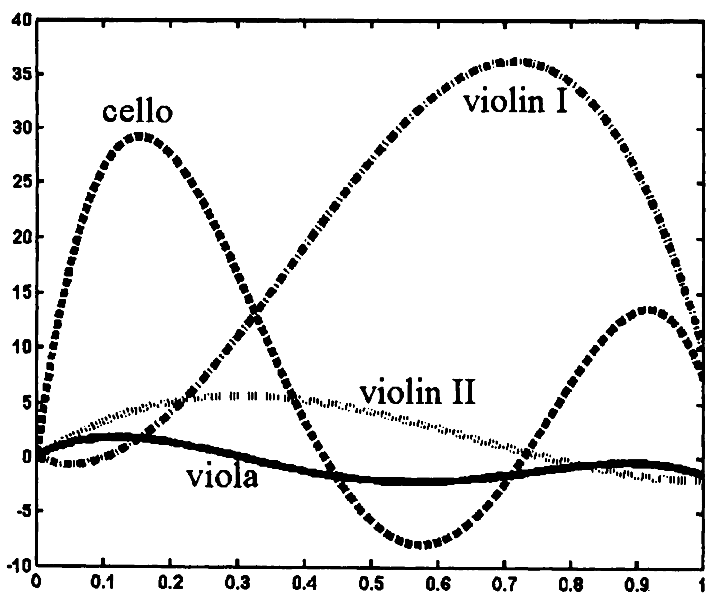


Example 19. Carter, String Quartet No. 2, III, mm. 300-303: (a) musical example, and (b) trace graphs.

(a)



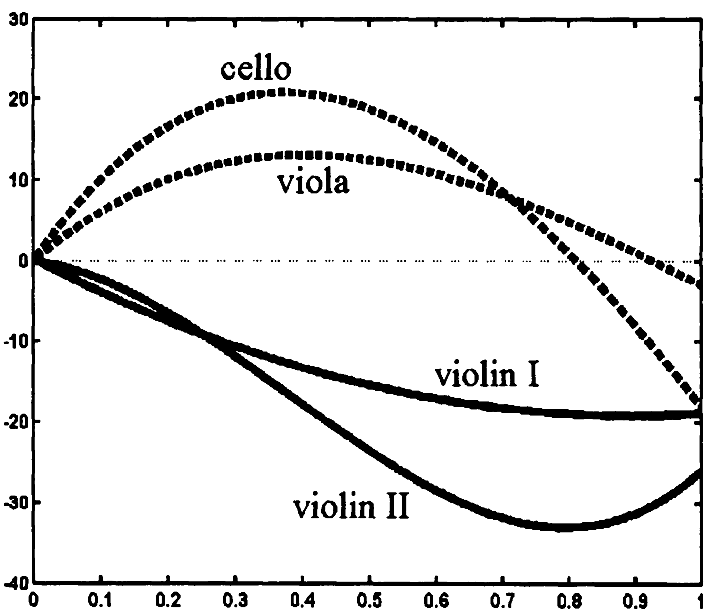
(b)



Example 20. Carter, String Quartet No. 2, III, mm. 338-340: (a) musical example, and (b) trace graphs.

(a)

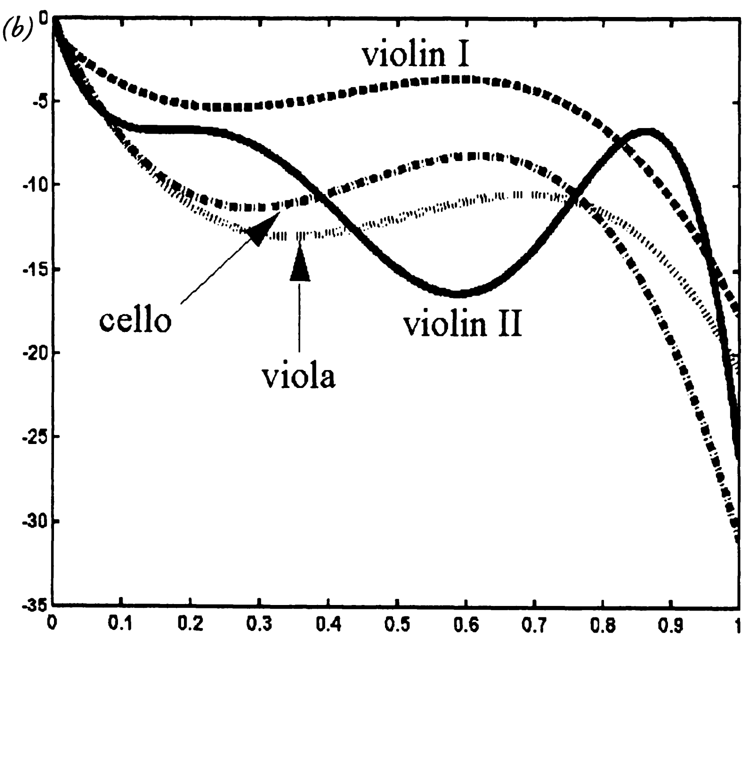
(b)



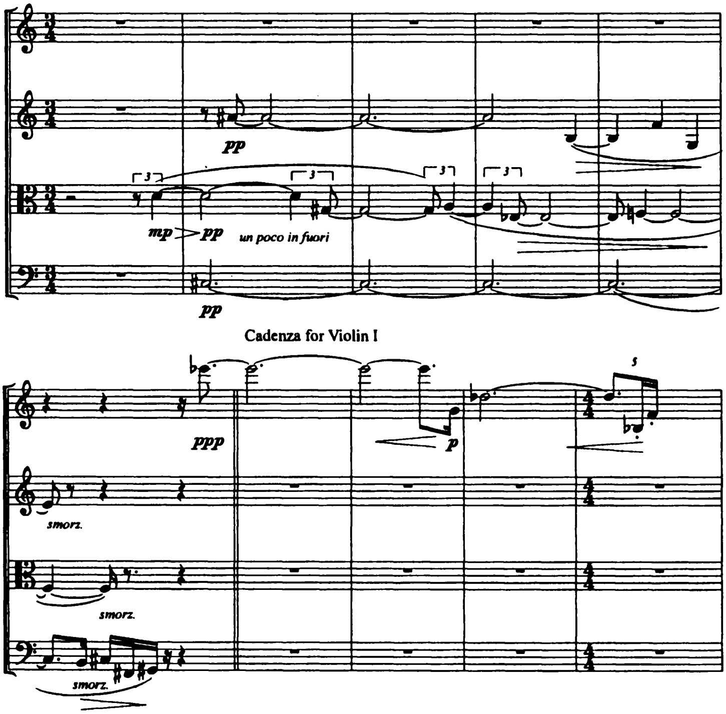
Example 21. Carter, String Quartet No. 2, III, mm. 355-358:
(a) musical example, and (b) trace graphs.

(a)

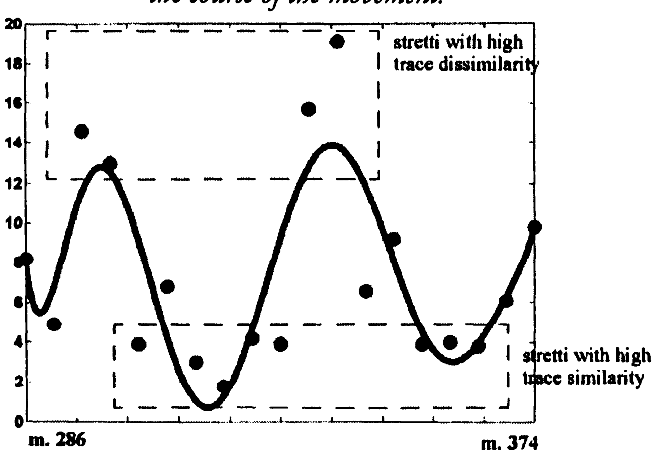
(b)



Example 22. Carter, String Quartet No. 2, III, mm. 368-377.



Example 23. Progression of the stretti's average integral distances over the course of the movement. *



*Note that the time-scale is not meant to be accurate; the x-axis spaces the stretti occurrences evenly, merely showing a progression over the course of the movement.

The passage in mm. 338-340 (Example 20a) can be separated into two strata on the basis of trace similarities. The two violin fragments do not share a contour by the definitions of Morris and Friedmann, though they do have identical adjacent-note directions. In both parts, the melodic motion is downward-downward-upward. The viola and cello lines are also similar in adjacent-note direction, as the motion in each is upward-downward-downward. Not surprisingly, then, the similarity diagram (in Example 18l) groups the violin parts together on the basis of their integral distance of 7.2, and correspondingly groups the viola and cello parts, which are at an integral distance of 5.7. The integral distances separating members of the two groups are comparatively large, all greater than 20. This separation can be seen in the graphs of Example 20b. Though the two groups share some dynamics and articulations, their melodic traces differentiate them.

The passage in mm. 355-358 (see Example 21a) would also seem to separate into two strata. The violins each have a short first note, followed by three long notes in a generally descending direction, ending with a large leap downward in shorter rhythmic values. The viola and cello, on the other hand, each have a four-note figure of the same contour, mostly in long rhythmic values. However, in this case, the trace analysis reveals an underlying similarity among all four melodic fragments. The overall descending character of both groups outweighs the "surface" dissimilarities, resulting in a similarity diagram in which all four parts are relatively close, as measured by integral distance (see Example 18p).

The last of the nineteen stretti analyzed here straddles the border of the third movement proper and the first violin cadenza (see Example 22). The violin II and viola parts exhibit a remarkable trace similarity (d_I = 1.9), despite their different rhythmic profiles. The cello part, too, though its rhythm is even further distended by the C♯3 held for four full measures and the rapid sixteenth notes that follow it, maintains a general proximity to the second violin and viola parts, at an integral distance of 9.0 and 7.4, respectively. It is only the first violin part that vigorously breaks from the mold, at an average d_I of 13.5 from the other melodic fragments. See Example 18s for a similarity diagram of the passage. In examples such as this one, in which widelydifferentiated rhythmic values exist side-by-side in most of the parts, the trace analysis we constructed "smooths over" the large rhythmic differences. The integral distance more closely follows the differences in local pitch space among the parts. Thus, the similar pitch-space range of the three lower parts, coupled with their identical Morris/Friedmann csegs, holds their integral distances relatively close, despite the variations in rhythm. On the other hand, the first violin, which has a compass of 29 semitones, is judged by integral distance to be "farther" from the other parts, which each encompass 15 semitones or less.

Conclusions

The viola and the first violin have very prominent roles in this movement. In the nineteen stretti analyzed, the viola "leads" in all but two; that is, nearly every stretto begins with a note from the viola. Among the fourteen stretti involving the first violin, that part "trails" (is the last to stop playing) in all but three. Similarity diagrams can be used to identify the most dissimilar trace of each grouping, based on the relative sizes of the integral distances. For the sixteen stretti with more than two instrumental parts, the viola has the most dissimilar trace in seven, and the first violin is the most dissimilar in six.

The points in Example 23 represent the average integral distances of the nineteen analyzed stretti in the order that they appear. Since the average integral distance is a rough gauge of any particular stretto's cohesiveness—how tightly-bound by trace similarity the parts are—this graph models the shifting relationship among the four parts. A trace of this data has been derived using a least-squares approximation and has been graphed along with the points to aid visualization. Two peaks can easily be discerned in the figure, correlating to passages wherein the instrumental parts are least similar. Two troughs, corresponding to stretti in which the parts are in very close imitation (as measured by integral distance), are also visible. The movement begins with the four parts at a sort of "average" distance from each other, fluxuates between extreme dissimilarity and extreme similarity, and then returns to the average distance as the movement ends. The graph in Example 23 thus serves as a possible model for listening to and interpreting the movement as a whole, based on the unique properties of melodic trace analysis. It also exhibits a property that David Schiff attributes to the quartet as a whole: "two interlaced processes," one moving towards greater cooperation among the instruments, one moving towards greater opposition.[23]

The trace is a flexible, adaptable tool for representing relationships between musical parameters. As shown above, it can model structures like Carter's melodic fragments and quantify aspects of imitation that are difficult to analyze using established methods. In particular, it is possible to compare melodies with differing numbers of notes easily. Thoughtful application of trace methods can yield novel, and musically meaningful, insights.

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[23] See Schiff 1998, 73.

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